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ON LOCAL PROPERTIES OF CLOSED SETS

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The purpose of this paper is to give some applications of a fundamental notion introduced by Lefschetz in Analysis Situs—the notion of relative cycle. There is no branch in Topology in which this notion has not shown itself as one of the most fruitful. I believe, in particular, that relative cycles govern the whole theory of local invariants of point-sets and especially the theory of ramification of curves, surfaces and continua of higher dimensions. I intend to give here the first definitions and the first elementary theorems which might serve as an introduction to a general theory of local invariants based entirely on relative cycles. Some relations (concerning the one dimensional case) between this theory and the theory of Urysohn and Menger are given in §4.* A short exposition of the main results of this Memoir is given in my Notes "*Sur les propriétés locales des ensembles fermés*" and "*Les groupes de Betti en un point*" published in the *Comptes rendus hebdomadaires des séances de l'Académie des Sciences*, t. 198, January, 1934, 227–229, 315–317.

This Memoir is closely related to my papers:

1) "Untersuchungen über Gestalt und Lage abgeschlossener Mengen beliebiger Dimension," these Annals, (2), 30 (1929), 101–187 (referred to as "*Gestalt und Lage*");

2) "Dimensionstheorie," Math. Annalen 106 (1932), 161–238 ("*Dimensionstheorie*")

3) "Über die Urysohnschen Konstanten," Fund. Math. 20 (1933), 140–151 ("*Urysohnsche Konstanten*").

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* An attempt to generalize the Urysohn-Menger definition to the two dimensional case has been recently made by Mrs. Różańska.

† (Added in proof November 10, 1934.) There appeared a memoir of Čech "Sur les nombres de Betti locaux" (these Annals, vol. 35, (1934), p. 678) which has some points of contact with the present paper. In particular, Čech gives the definition of what I call Betti numbers around a point and proves the Duality Theorem of section 2.

1. The Betti numbers around a point

1. Let $F \subset R^n$ be a closed bounded point-set immersed in Euclidean space R^n of n dimensions. We consider an arbitrary point a of F. Let $U(a, \epsilon)$ denote the ϵ-neighborhood of the point a in the space R^n i.e. the interior of the sphere with the centre a and the radius ϵ (ϵ being any positive number).

We choose, now, two positive numbers ϵ and σ, $\epsilon \geqq \sigma$, and denote by $q^s_{\sigma,\epsilon}$ the maximal number of s-dimensional (absolute) cycles situated in $U(a, \sigma) - F$ and independent with respect to homologies in $U(a, \epsilon) - F$. Any rational numbers[1] are allowed as coefficients in cycles as well as in homologies.

When σ decreases, $q^s_{\sigma,\epsilon}$ either decreases or remains constant so that there exists

$$\lim_{\sigma \to 0} q^s_{\sigma,\epsilon} = q^s_\epsilon .$$

When ϵ decreases, q^s_ϵ cannot decrease. There exists, therefore, a finite or infinite limit

$$q^s(a, R^n - F) = \lim_{\epsilon \to 0} q^s_\epsilon .$$

The number $q = q^s(a, R^n - F)$ will be called the *s-dimensional Betti number of $R^n - F$ around the point a.*

REMARK. In the case $s = 0$ we consider in $R^n - F$ only cycles with a vanishing sum of coefficients. Cf. *Alexander*, Trans. Amer. Math. Soc., 23 (1922), 233–349.

2. A little reflection makes it clear how natural this definition is; $q^0(R^n - F)$ gives us in particular the number of domains into which F decomposes the space in the neighborhood of its point a.

Now the problem is to give to the number $q^s(a, R^n - F)$ a characterization, which is independent of any consideration of $R^n - F$, so that in other words, we have $q^s(R^n - F)$ as a local invariant of the set F *itself*. It is for this purpose that we need the Lefschetz relative cycles!

Let us consider again two arbitrary positive numbers ϵ and $\sigma < \epsilon$. The number of r-dimensional cycles on $\overline{F \cdot U(a, \epsilon)}$ modulo $[F - U(a, \epsilon)]$ independent with respect to homologies in $F \cdot \overline{U(a, \epsilon)}$ modulo $[F - U(a, \sigma)]$ will be called $p^r_{\epsilon,\sigma}$ for every non-negative r. From $\sigma > \sigma'$ it follows at once that $p^r_{\epsilon,\sigma} \geqq p^r_{\epsilon,\sigma'}$; therefore

$$\lim_{\sigma \to 0} p^r_{\epsilon,\sigma} = p^r_\epsilon$$

exists. If $\epsilon > \epsilon'$, then $p^r_\epsilon \leqq p^r_{\epsilon'}$.

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[1] In other words, we have the field of rational numbers as *Koeffizientenbereich*. Concerning this general notion see *Urysohnsche Konstanten*. The field of rational numbers was introduced as a Koeffizentenbereich by Lefschetz in his fundamental paper "Closed point-sets on a manifold," these Annals, 1929, 232–254.

To prove this assertion let us consider an integer number k which is by definition equal to p_ϵ^r, if the last number is finite, and which is arbitrarily large if p_ϵ^r is infinite. We have only to show that for an arbitrarily chosen $\epsilon' < \epsilon$ and $\sigma < \epsilon'$ there are at least k cycles on $F \cdot \overline{U(a, \epsilon')}$ mod $[F - U(a, \epsilon')]$ independent mod $[F - U(a, \sigma)]$. Now there are at least k cycles $Z_1, Z_2, \cdots, Z_k$ on $F \cdot \overline{U(a, \epsilon)}$ mod $[F - U(a, \epsilon)]$ independent mod $[F - U(a, \sigma)]$. If we take from each of these cycles its part lying on $\overline{U(a, \epsilon')}$, we get cycles $Z_1' = Z_1 \cdot \overline{U(a, \epsilon')}$, $Z_2' = Z_2 \cdot \overline{U(a, \epsilon')}, \cdots, Z_k' = Z_k \cdot \overline{U(a, \epsilon')}$ mod $[F - U(a, \epsilon')]$ and these are independent mod $[F - U(a, \sigma)]$. In fact, if

$$\sum t^i Z_i' \sim 0 \qquad \text{mod } [F - U(a, \sigma)],$$

then

$$\sum t^i Z_i = \sum t^i (Z_i - Z_i') + \sum t^i Z_i' \sim \sum t^i (Z_i - Z_i') \qquad \text{mod } [F - U(a, \sigma)],$$

which means

$$\sum t^i Z_i \sim 0 \qquad \text{mod } [F - U(a, \sigma)],$$

because $\sum t^i (Z_i - Z_i')$ is on $F - U(a, \epsilon') \subset F - U(a, \sigma)$ and therefore is ~ 0 mod $[F - U(a, \sigma)]$.

From what we have just proved, follows the existence of a finite or infinite limit:

$$\lim_{\epsilon \to 0} p_\epsilon^r = p^r(a, F).$$

This limit will be called the *r-dimensional Betti number of F around the point a.*

REMARK. There are two cases to be distinguished if $p^r(a, F) = \infty$. In the first case p_ϵ^r is finite for an arbitrarily small ϵ, but $\lim_{\epsilon \to 0} p_\epsilon^r = \infty$. In this case we write $p^r(a, F) = \omega$, and say that $p^r(a, F)$ is *increasingly infinite.* In the second case there exists a certain ϵ for which $p_\epsilon^r = \infty$; as ϵ decreases, p^r still remains infinite and we use the notation $p^r(a, F) = \infty$. We then say that $p^r(a, F)$ is *actually infinite.*

3. EXAMPLES. I. For all interior points of a straight line segment we have $p^1(a) = 1$, while for the end-points the corresponding number is equal to zero. In the same way the one-dimensional local Betti number is equal to 1 around every point of a circumference.

II. Around every point of a closed surface (in the classical sense) the first Betti number is equal to zero, while the second is equal to 1.

If F is a square, then around a boundary point both the first and the second local Betti numbers vanish; around an interior point the first number vanishes, while the second is equal to 1.

III. Let F be a double cone, a its vertex. Then $p^1(a, F) = 1$, $p^2(F) = 2$.

IV. Let F be the point set formed by two squares having a straight line seg-

ment s as their intersection. For an interior point a of s we have $p^2(a, F) = 3$, $p^1(a, F) = 0$.

In all these examples $p^0(a, F)$ is equal to 0.

V. If F consists of a point a and the circumferences having a for their common centre and the numbers $1, \frac{1}{2}, \frac{1}{3}, \cdots$ for their radii, then $p^0(a, F) = p^1(a, F) = \infty$.

VI. Let F consist of all the points $x = 0$, $-1 \leqq y \leqq 1$ and the points $0 < x \leqq 1/\pi$, $y = \sin 1/x$ in the (x, y)-plane. For every point a of F situated on the y-axis $p^1(a, F)$ is infinite.

REMARK. We have chosen the field of all rational numbers as the *field of coefficients* (Koeffizientenbereich). We need scarcely say that every other Abelian group would lead to analogous definitions. One has only to define properly the notion of linear independence (with respect to homology and to the chosen field of coefficients). We can in particular speak about the Betti numbers modulo 2 around a point.

2. The duality formula

1. We shall now give the proof of the following

DUALITY THEOREM. *$p^r(a, F) = q^{n-r-1}(a, R^n - F)$ for every point a of the closed set $F \subset R^n$.*

PROOF: We write $Z(\epsilon, \sigma)$ for an r-dimensional cycle of $F \cdot \overline{U(a, \epsilon)}$ mod $[F - U(a, \epsilon)]$, which does not bound mod $[F - U(a, \sigma)]$; moreover we denote by $z(\sigma, \epsilon)$ an absolute $(n - r - 1)$-dimensional cycle lying in $U(a, \sigma) - F$ and not homologous to zero in $U(a, \epsilon) - F$. Then the duality theorem follows after an elementary matrix consideration from two lemmas:

LEMMA 1. *For each $Z(\epsilon, \sigma)$ there exists one $z(\sigma, \epsilon)$ linked[2] with $Z(\epsilon, \sigma)$.*

LEMMA 2. *For each $z(\sigma, \epsilon)$ there exists a $Z(\epsilon, \sigma)$ linked with $z(\sigma, \epsilon)$.*

2. In order to prove Lemma 1 we can proceed as follows. First let $r < n - 1$ and

$$Z(\epsilon, \sigma) = (z_1, z_2, \cdots, z_k, \cdots)$$

where z_k is, as usual, a combinatorial (polyhedral) ϵ_k-cycle mod $[F - U(a, \epsilon)]$ and $\lim \epsilon_k = 0$; let c_k be a certain ϵ_k'-complex, $\lim \epsilon_k' = 0$, bounded by the boundary $\dot{z}_k$ of z_k and lying on $S^{n-1} = \overline{U}(a, \epsilon) - U(a, \epsilon)$. Then $z_k - c_k = \hat{z}_k$ is an ϵ_k''-cycle in $F'' = F \cdot U(a, \epsilon) + [\overline{U}(a, \epsilon) - U(a, \sigma)]$, $\lim \epsilon_k'' = 0$ and it is easy to see that

$$\hat{Z} = (\hat{z}_1, \hat{z}_2, \cdots, \hat{z}_k, \cdots)$$

is an absolute cycle on F''.

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[2] We say, that $Z(\epsilon, \sigma)$ and $z(\sigma, \epsilon)$ are linked (in $U(a, \epsilon)$) if the intersection number of $Z(\epsilon, \sigma)$ and any complex lying in $U(a, \epsilon)$ and bounded by $z(\sigma, \epsilon)$ is different from zero. See Lefschetz "Closed point-sets on a manifold," Annals of Math. (2) 1929, 232–254 and *Dimensionstheorie*, especially §2.

The only thing to be proved is the "convergence" of $\hat{Z}$: we must show in fact that for every ϵ there exists a certain n such that for $k > n$ the homology $\hat{z}_n \underset{\epsilon}{\sim} \hat{z}_k$ holds in F'' with respect to the field of rational numbers as the coefficients-field. Now from the convergence of $Z(\epsilon, \sigma)$ it follows that ϵ_k'''-complexes $C_{k,h}$ exist with

$$\dot{C}_{k,h} = (z_k - z_{k+h}) + Q_{k,h}, \qquad Q_{k,h} \subset S^{n-1}$$

(we use $\dot{C}$ for the boundary of C). $\dot{C}_{k,h}$ being an absolute cycle, we have

$$\dot{Q}_{k,h} = -(z_k - z_{k+h})^{\cdot} = -(c_k - c_{k+h})^{\cdot}.$$

The complex $c_k - c_{k+h} + Q_{k,h}$ is therefore a cycle on S^{n-1} and it bounds there a certain $(r+1)$-dimensional complex $U_{k,h}$:

$$\dot{U}_{k,h} = c_k - c_{k+h} + Q_{k,h}.$$

This leads us to

$$(C_{k,h} - U_{k,h})^{\cdot} = z_k - z_{k+h} + Q_{k,h} - Q_{k,h} - c_k + c_{k+h} = (z_k - c_k) - (z_{k+h} - c_{k+h}).$$

$C_{k,h} - U_{k,h}$ being an ϵ_k^{IV}-complex in F'', we arrive at exactly what we want to prove.

$\hat{Z}$ does not bound in F''. Let us suppose, in fact, that there exist ϵ_k-complexes C_k in F'' with $\dot{C}_k = \hat{z}_k$. We put $\hat{C}_k = C_k \cdot \bar{U}(a, \sigma)$ and have:

$$\dot{\hat{C}}_k = \dot{C}_k - (C_k - \hat{C}_k)^{\cdot} = \hat{z}_k - (C_k - \hat{C}_k)^{\cdot}$$

$$= z_k + (\hat{z}_k - z_k) - (C_k - \hat{C}_k)^{\cdot} = z_k \pmod{[F - U(a, \sigma)]}$$

which contradicts the definition of z_k.

We take now any cycle z lying in $R^n - F''$ and linked with $\hat{Z}$; the existence of such a cycle follows, for instance, from the 1. Verschlingungssatz given in *Dimensionstheorie* (p. 184). This cycle z has a positive distance from $\overline{U(a, \epsilon)} - U(a, \sigma)$ and can be decomposed into the sum of two cycles: $z = \hat{z} + \overset{*}{z}$, one of them say $\hat{z}$, lying in $U(a, \sigma)$, the other belonging to $R^n - \overline{U}(a, \epsilon)$. The cycle $\hat{z}$ is, of course, linked with $\hat{Z}$ and also with Z, which proves Lemma 1 in case $r < n - 1$.

3. The only difficulty which arises in case $r = n - 1$ is that now z is a zero-dimensional cycle. The decomposition $z = \hat{z} + \overset{*}{z}$ holds in this case also, but it may happen, that the sum of the coefficients in $\hat{z}$ is different from zero; zero-dimensional cycles of this kind are not allowed in the present treatment.

To save the situation let us consider the sequence

$$Z' = (z_1', z_2', \cdots, z_k', \cdots)$$

of successive (barycentric) subdivisions of the sphere S^{n-1} as an $(n-1)$-dimensional cycle of the point-set F''. We shall first prove that the homology

$$\hat{z}_k \underset{\epsilon_k}{\sim} t_k z_k' \qquad \text{(on } F'') \tag{1}$$

cannot hold with $\lim \epsilon_k = 0$. With the opposite hypothesis we have in fact n-dimensional ϵ_k-complexes $\hat{c}_k^n \subset F''$ with

$$\dot{\hat{c}}_k^n = \hat{z}_k^n - t_k z_k' = z_k - c_k - t_k z_k',$$

which gives us after the decomposition $\hat{c}_k^n = c_k^n + c_k'$ with $c_k^n \subset F \cdot U(a, \sigma)$, $c_k' \subset \overline{U}(a, \epsilon) - U(a, \sigma)$

$$\dot{c}_k^n = \dot{\hat{c}}_k^n - \dot{c}_k' = z_k - (c_k + t_k z_k' + \dot{c}_k').$$

Since $(c_k + t_k z_k' + c_k')$ belongs to $\overline{U(a, \epsilon)} - U(a, \sigma)$, the last relation means that z_k bounds in F mod $[F - U(a, \sigma)]$ the ϵ_k-complex c_k^n, which is again a contradiction.

The homology (1) being impossible, we can always find a cycle $\hat{z}$ linking $\hat{Z}$, but not Z' and which is the boundary of a (one-dimensional) complex C; this implies that $\hat{z}$ has a vanishing sum of coefficients. We have then $C = C_1 + C_2$, where C_1 lies in $U(a, \sigma) - F$; the boundary of C_1 is a zero-dimensional complex with a vanishing sum of coefficients which links $Z(\epsilon, \sigma)$.

Lemma 1 is now completely proved.

4. For the proof of Lemma 2, it is sufficient to apply the remark concerning relative cycles which I gave in my above mentioned "Dimensionstheorie" under the name of "*5. Verschlingungssatz*" (p. 191). It follows from that remark that there exists a cycle mod $[F - U(a, \epsilon)]$, say Z, which links the given $z(\sigma, \epsilon)$. Since $z(\sigma, \epsilon)$ is within $U(a, \sigma)$, the relative cycle Z cannot bound mod $[F - U(a, \sigma)]$, it is therefore a $Z(\epsilon, \sigma)$, q.e.d.

After the Lemmas are proved, we have only to apply the argument used in *Gestalt und Lage*, pp. 164–167.

This leads us, first of all, to the notions of an *r-dimensional basis of F around a with respect to σ and ϵ* and to an analogous basis of $R^n - F$. The first of these bases will be denoted by $\mathfrak{B}^r_{\epsilon,\sigma}(a, F)$, the second by $\mathfrak{B}^s_{\sigma,\epsilon}(a, R^n - F)$, s being in our case equal to $n - r - 1$. They are defined as follows.

A set of cycles $Z(\epsilon, \sigma)$, finite or enumerable, is called a $\mathfrak{B}^r_{\epsilon,\sigma}(a, F)$, if the two following conditions are fulfilled:

1. Given any r-dimensional cycle mod $[F - U(a, \epsilon)]$ and any positive number δ, this cycle is δ-homologous mod $[F - U(a, \sigma)]$ to a certain (finite) linear combination of the cycles of the given set.

2. A (finite) linear combination of cycles of the given set bounds on F mod $[F - U(a, \sigma)]$ only if the coefficients of members of this combination vanish.

In an analogous way a set of cycles $z(\sigma, \epsilon)$ is called a $\mathfrak{B}^s_{\sigma,\epsilon}(a, R^n - F)$ if the following conditions hold:

1′. Any s-dimensional (absolute) cycle on $U(a, \sigma) - F$, is homologous on $U(a, \epsilon) - F$ to a (finite) linear combination of the given cycles.

2′. A (finite) linear combination of cycles of the given set bound on $U(a, \epsilon) - F$ only if all the coefficients involved vanish.

Using exactly the same procedure as in my paper cited above we prove that

$\mathfrak{B}^r_{\epsilon,\sigma}(a, F)$ and $\mathfrak{B}^{n-r-1}_{\sigma,\epsilon}(a, R^n - F)$ can always be chosen in such a manner that each element of the one basis is linked with one and only one element of the other, the linking coefficient being equal to 1. (We say in this case that the two bases are *linked.*) After that, it follows by an elementary algebraic argument that

$$q^{n-r-1}_{\sigma,\epsilon}(a, R^n - F) = p^r_{\epsilon,\sigma}(a, F),$$

which includes our duality formula.

REMARK. All this holds also modulo m with the only modification that if m is not a prime, the linking coefficients may be different from 1.

COROLLARY I. *The Betti numbers of $R^n - F$ around the different points of F are invariant with respect to topological transformations of F.*

In other words: if F and F' are homeomorphic point-sets in R^n and if a and a' correspond to each other in a homeomorphism between F and F', then $q^s(a, R^n - F) = q^s(a', R^n - F')$ for every s.[4] In particular for $r = 0$ we get: *If F decomposes the space around the point a in k components, then F' has the same property around the point a'.*

COROLLARY[5] II. *Let E^r be an r-dimensional cell (i.e. the topological image of an r-dimensional simplex) immersed in the n-dimensional space R^n. Then*

$$q^{n-r-1}(a, R^n - E^r) = 1,$$

if a is an interior point of E^r; otherwise $q^{n-r-1}(a, R^n - E^r) = 0$. If $s \neq n - r - 1$ and a is an arbitrary point of E^r then $q^s(a, R^n - E^r) = 0$.

SKETCH OF THE PROOF. Using our duality formula we can first replace q by the corresponding p. Furthermore by the standard process of small deformations, we reduce every $Z(\epsilon, \sigma)$ to a cycle whose elements z_k are sub-complexes of successive subdivisions of E^r. After this transformation to the combinatorial case, the proof presents no difficulty at all.

3. Local Connectedness

1. There are various ways to introduce local connectedness in n dimensions. We shall consider here the definition given in 1928 in my paper *Gestalt und Lage.*[6] According to this definition F is called *locally connected at its point a with respect to the dimensionality r and the field of coefficients $\mathfrak{F}$*, if to each $\epsilon > 0$ belongs a $\sigma > 0$ such that an arbitrary r-dimensional (absolute) cycle (belonging to the field of coefficients $\mathfrak{F}$) on the σ-neighborhood of a (on F) bounds within the ϵ-neighborhood of the same point.

In this section we shall consider only the set of all rational numbers as the

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[4] If $F' \subset R^{n'}$ then $q^s(a, R^n - F) = qs'(a', R^{n'} - F')$ with $s' = s + n' - n$.

[5] This corollary gives the answer to a question proposed to me by Hopf in his letter of December 15th, 1928.

[6] P. 181, footnote [63]. As pointed out loc. cit., the same definition was independently given by Alexander, who, however, never published it (so far as I know). Another definition (which is materially different from ours) is given by Lefschetz, Topology.

field of coefficients. As a matter of fact, the following considerations hold also modulo m, m being an arbitrary integer $\geqq 2$.

2. Our purpose is to show what happens if F is not locally connected (with respect to a fixed r) at its point a. We shall see that there is a complete analogy between our present case and the subject matter of the previous section.

We denote for a moment by $Z[\sigma, \epsilon]$ an r-dimensional *absolute* cycle on $F \cdot \overline{U(a, \sigma)}$ which does not bound on $F \cdot \overline{U(a, \epsilon)}$. In an analogous way let $z[\epsilon, \sigma]$ be an $(n - r - 1)$-dimensional cycle on $U(a, \epsilon) - F$ mod $[R^n - U(a, \epsilon)]$ which does not bound on $R^n - F$ mod $R^n - U(a, \sigma)$. We prove now:

LEMMA 1. *There exists for each $Z[\sigma, \epsilon]$, a $z[\epsilon, \sigma]$ linked with it.*[7]

LEMMA 2. *There exists for each $z[\epsilon, \sigma]$, a $Z[\sigma, \epsilon]$ linked with it.*

PROOF OF LEMMA 1. As $Z[\sigma, \epsilon]$ does not bound on $F \cdot \overline{U(a, \epsilon)}$, there exists on $R^n - \overline{F \cdot U(a, \epsilon)}$ an $(n - r - 1)$-dimensional absolute cycle z' linked with $Z[\sigma, \epsilon]$. The part z of z' lying in $U(a, \epsilon)$ is a cycle mod $[R^n - U(a, \epsilon)]$ linking the given $Z[\sigma, \epsilon]$ and therefore not bounding mod $[R^n - U(a, \sigma)]$; thus it forms the desired $z(\epsilon, \sigma)$.

PROOF OF LEMMA 2. We put $F' = F \cdot \overline{U(a, \epsilon)}$ and choose a positive number δ which is so small that the given $z[\epsilon, \sigma]$ is on $\overline{U(a, \epsilon)} - U(F', \delta)$. Then we define such a number δ' that every cycle on $\overline{U(F', \delta')}$ is homologous within $U(F', \delta)$ to a cycle on F'. The existence of such a δ' follows from well known results due to Lefschetz.[8]

Now we consider $z[\epsilon, \sigma] \cdot U(a, \sigma)$ as a relative cycle on the closed set $F'' = \overline{U(a, \sigma)} - U(F', \delta')$. Using the same argument as in the proof of Lemma 1 of the previous section, we find an absolute cycle z' on

$$U(a, \sigma) - F'' \subset U(F', \delta') \cdot U(a, \sigma)$$

linked with $z[\epsilon, \sigma]$. By definition of δ' there exists an absolute cycle z on $F' \cdot U(a, \sigma)$ homologous to z' within $U(F', \delta)$, and therefore also linked with $z(\epsilon, \sigma)$. The cycle z is a cycle $Z[\sigma, \epsilon]$ as we wished to have it.

3. We are now lead to introduce the numbers

$$\pi^r(a, F) = \lim_{\epsilon} \lim_{\sigma} \pi^r_{\sigma, \epsilon}, \qquad \kappa^s(a, R^n - F) = \lim_{\epsilon} \lim_{\sigma} \kappa^s_{\epsilon, \sigma},$$

where $\pi^r_{\sigma, \epsilon}$ is the maximum number of $Z[\sigma, \epsilon]$ independent with respect to homologies on $F \cdot \overline{U(a, \epsilon)}$, and $\kappa^s_{\epsilon, \sigma}$ the maximum number of $z[\epsilon, \sigma]$ $(s = n - r - 1)$ independent with respect to homologies on $R^n - F$ mod $[R^n - U(a, \sigma)]$.

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[7] It follows from the previous section, that in case $Z[\sigma, \epsilon]$ does not bound on $F \cdot \overline{U(a, \epsilon)}$ mod $[F - U(a, \sigma)]$ we can suppose that $z[\epsilon, \sigma]$ is an absolute cycle.

[8] Lefschetz, Annals of Math. (2) 29 (1928), 232–254.

We have then the *duality formula*

$$\pi^r(a, F) = \kappa^{n-r-1}(a, R^n - F).$$

In fact we may state more definitely:

THEOREM. *Let F be any closed set immersed in R^n, a a point of F, r an integer, $0 < r < n - 1$. Then, two cases only are possible: either F is locally connected at the point a (with respect to r) and*

$$\pi^r(a, F) = \kappa^{n-r-1}(a, R^n - F) = 0$$

or

$$\pi^r(a, F) = \kappa^{n-r-1}(a, R^n - F) = \infty.$$

PROOF. We have only to show that if $p^r_{\sigma,\epsilon}$ is finite, then for a sufficiently small σ_1' every cycle on $F \cdot \overline{U(a, \sigma_1')}$ bounds on $F \cdot \overline{U(a, \epsilon)}$. This follows from

LEMMA 3. *Let $\pi^r_{\sigma,\epsilon}(a, F)$ be finite. Then there exists a $\sigma_1 > 0$ such that an arbitrary r-dimensional cycle on $F \cdot \overline{U(a, \sigma_1)}$ is homologous on $F \cdot \overline{U(a, \epsilon)}$ to a cycle on $F \cdot \overline{U(a, \sigma')}$, for every $\sigma' < \sigma_1$.*

PROOF OF LEMMA 3. As $\pi^r_{\sigma,\epsilon}$ can only decrease when σ decreases, σ_1 can be chosen in such a manner that for every $\sigma' < \sigma_1$

$$\pi^r_{\sigma',\epsilon} = \pi^r_{\sigma_1,\epsilon} = p.$$

Let $z_1, \cdots, z_p$ be a basis of the $Z[\sigma_1, \epsilon]$, and $z', \cdots, z_p'$ an analogous basis of the $Z[\sigma', \epsilon]$; if there is an element—say z_1—of the first basis which is independent (with respect to homologies on $F \cdot \overline{U(a, \epsilon)}$) from the elements of the second basis, we should obtain a system of $p + 1$ cycles $z_1, z_1', \cdots, z_p'$ on $F \cdot \overline{U(a, \sigma_1)}$, independent on $F \cdot \overline{U(a, \epsilon)}$, which is obviously impossible.

Thus our theorem is proved.

4. EXAMPLES. 1. F is the curve $y = \sin 1/x$ with the segment $-1 \leqq y \leqq 1$ of the y-axis (§1, Nr. 3 ex. VI). If a is a point $(0, y)$ with $-1 < y < 1$ then $p^0(a, F) = \pi^1(a, F) = 0$, while $\pi^0(a, F) = p^1(a, F) = \infty$.

2. F consists of the point $a = (0, 0)$ of the (xy)-plane and of all points of the circumferences

$$(x - 3/2^n)^2 + y^2 = 1/4^n, \qquad n = 1, 2, \cdots,$$

$$p^0(a, F) = \pi^0(a, F) = 0; \qquad p^1(a, F) = \pi^1(a, F) = \infty.$$

3. F consists of the points $(0, 0, z)$, $0 \leqq z \leqq 1$ of the (xyz)-space and of all points of the same space which satisfy the relations:

$$0 \leqq z \leqq 1, \qquad (x - 3/2^n)^2 + y^2 = 1/4^n, \qquad n = 1, 2, \cdots.$$

If a is a point $(0, 0, z)$ with $0 < z < 1$, then

$$p^0(a, F) = \pi^0(a, F) = 0$$

$$p^1(a, F) = 0; \qquad \pi^1(a, F) = \infty$$

$$p^2(a, F) = \infty; \qquad \pi^2(a, F) = 0.$$

4. The one-dimensional case

1. LEMMA 1. *Let F be an r-dimensional closed point-set of the n-dimensional space R^n. Every s-dimensional cycle, $s < n - r - 1$, situated in $R^n - F$ is in $R^n - F$ homologous to an algebraic sum of cycles whose diameters are less than an arbitrary positive number.*

PROOF. The given s-dimensional cycle $z \subset R^n - F$ bounds a certain complex C. We can obviously suppose that the simplexes of C are less than the arbitrary small positive number ϵ. Let us choose the positive number σ so small that the diameters of the simplexes of C are still less than $\epsilon - \sigma$. Applying the σ-modification process[9] to the simplexes of C of dimensionalities $\leqq n - r - 1$ we can replace C by a complex C' still bounded by z and whose $(n - r - 1)$-dimensional elements (which may not be simplexes) have no points in common with F. According to our choice of σ, the $(n - r)$-dimensional elements of C' can still be supposed to be less than ϵ. After this z is homologous in $R^n - F$ to the sum of the boundaries of those $(s + 1)$-dimensional elements of C' which have points in common with F. Lemma 1 is therefore proved.

LEMMA 2. *If there is on the one-dimensional continuum K, an infinity of subcontinua $K_1, K_2, \cdots, K_i, \cdots$ no two of which have points in common and whose diameters are greater than a fixed positive number d, then K has, around certain of its points a, an actually infinite one-dimensional Betti number $p^1(a, K)$.*

REMARK. In this section we always understand by a Betti number of F around a the one-dimensional Betti number $p^1(a, F)$.

By the Menger immersion theorem, K can be supposed to be a continuum of R^3. Thus we need only to prove that for certain points a of K we must have $q^1(a, R^3 - K) = \infty$. We shall arrive at a contradiction of the hypothesis, that for every point a of K the number $q^1(a, R^3 - K)$ is finite or increasingly infinite.

By a well known convergence theorem[10] we can suppose that the sequence $K_1, K_2, \cdots$ converges to a continuum K_ω on K. If p and q are two arbitrary points of K, then points p_i and q_i can be chosen on K_i so that $\lim p_i = p$, $\lim q_i = q$. We can finally suppose that all of the p_i are included in $U(p, \sigma)$, all of the q_i in $U(q, \sigma)$, where 3σ is less than the distance between p and q. There exists among the components of $K_i - (U(p, \sigma) + U(q, \sigma))$ at least one, say K_1', which has points on the boundaries of $U(p, \sigma)$ and $U(q, \sigma)$; then K_i' contains a one-dimensional cycle $Z_i \bmod [\overline{U(p, \sigma)} + \overline{U(q, \sigma)}]$:

$$Z_i = (z_1^{(i)}, z_2^{(i)}, \cdots, z_k^{(i)}, \cdots),$$

where $z_k^{(i)}$ is a one-dimensional ϵ_k-cycle, $\lim \epsilon_k = 0$, $\bmod [\overline{U(p, \sigma)} + \overline{U(q, \sigma)}]$; this is, as a matter of fact, a finite sequence of points of K_i' joining a point a_i lying on the boundary of $U(p, \sigma)$ with a point b_i lying on the boundary of $U(q, \sigma)$. Among the components of $K_\omega - (U(p, \sigma) + U(q, \sigma))$ there is also at

[9] *Dimensionstheorie*, p. 204.

[10] See e.g. Hausdorff, Mengenlehre, p. 150, VI.

least one, say K'_ω, which has a point a_ω on the boundary of $U(p, \sigma)$ and a point b_ω on the boundary of $U(q, \sigma)$. We can choose the cycle

$$Z_\omega = (z_1^{(\omega)}, z_2^{(\omega)}, \cdots, z_k^{(\omega)}, \cdots) \bmod [\overline{U(p, \sigma)} + \overline{U(q, \sigma)}]$$

so that all $z_k^{(\omega)}$ have the same end-points a_ω and b_ω. If we finally join a_ω with a_k, b_ω with b_k by straight-line segments (lying respectively in $\overline{U(p, \sigma)}$ and $\overline{U(q, \sigma)}$), we get absolute cycles

$$Z'_i = (\overset{*}{z}_1^{(i)}, \overset{*}{z}_2^{(i)}, \cdots, \overset{*}{z}_k^{(i)}, \cdots),$$

where $\overset{*}{z}_k^{(i)} = \overline{a_\omega a_i} + z_k^{(i)} + \overline{b_i b_\omega} - z_k^\omega$. These cycles are obviously independent in $K + \overline{U(p, \sigma)} + \overline{U(q, \sigma)}$; they are also independent in $K + \overline{U(p, \sigma')} + \overline{U(q, \sigma')}$, where σ' is sufficiently small but greater than σ. We can therefore find cycles z_i in $R^3 - (K + \overline{U(p, \sigma')} + \overline{U(q, \sigma')})$ linking the corresponding Z'_i and accordingly independent in $R^3 - (K + \overline{U(p, \sigma)} + \overline{U(q, \sigma)})$.

It follows now from the supposed finiteness of $q^1(a, R^3 - K)$ that there exists for every point a of K a positive number σ_a such that there is only a finite number of one-dimensional cycles on $U(a, \sigma_a)$ which could be independent in $R^3 - (K + \overline{U(p, \sigma)} + \overline{U(q, \sigma)})$. We shall consider only points a of $K - (U(p, \sigma) + U(q, \sigma))$ and apply the Borel-Lebesgue theorem; we get in this way a finite number of points $a_1, a_2, \cdots, a_u$ such that

$$K - (U(p, \sigma) + U(q, \sigma)) \subset \Sigma\, U(a_i, \sigma_{a_i}).$$

We choose now the positive number δ in such a way that every cycle not bounding in $R^3 - (K + \overline{U(p, \sigma')} + \overline{U(q, \sigma')})$ lies in a certain $U(a_i, \sigma_{a_i})$ if it has a diameter less than δ.

By Lemma 1 at the beginning of this section we can suppose that the cycles z_i (linked with the Z'_i) being less than δ are therefore lying each in a certain $U(a_i, \sigma_{a_i})$. Thus among these cycles there can be only a finite number of independent ones in $R^3 - [K + U(p, \sigma) + U(q, \sigma)]$, which contradicts the definition of z_i. Lemma 2 is therefore proved.

2. It follows at once from what we have proved that a one-dimensional continuum is necessarily locally connected, if it has around no point an actually infinite Betti number. Such a continuum is therefore a *tree im kleinen*.[11]

It can be easily shown that a tree im kleinen is a *continuous curve of finite connection* (as defined in the memoir of the author "Über kombinatorische Eigenschaften allgemeiner Kurven, Math. Ann. 96 (1926), pp. 512–554, espe-

[11] A tree is, according to Mazurkiewicz, a locally connected continuum which has no simple closed curve as sub-continuum; it follows from this that every tree is one-dimensional. A tree *im kleinen* is a continuum, every point of which has a neighborhood whose closure is a tree. See Menger, Über reguläre Baumkurven, Math. Ann. 96 (1926), pp. 572–582.

cially Chapter V). In fact if K is a continuous curve of infinite connection, then an infinite sequence of simple closed curves $J_1, J_2, \cdots, J_k, \cdots$ can be found on K such that there are points on J_{k+1} which do not belong to $J_1, \cdots, J_k$ for every k. Therefore we have on K an infinite sequence of points

$$a_1, a_2, \cdots, a_k, \cdots$$

where a_k belong to J_k but not to J_{k-1}. If a is a limit-point of this sequence, then every neighborhood $U(a)$ of a must have points in common with an infinity of closed curves, all of which are different from each other *in* $U(a)$. This situation is impossible if $\overline{U(a)}$ is a tree.

Thus we have:

THEOREM. *The one-dimensional continua which have around each point a finite or increasingly infinite Betti number are identical with the continuous curves of finite connection (trees im kleinen). The Urysohn-Menger ramification index* $\mathrm{ind}_a K$ *of such a curve* K *in its point is given by the relation*

$$\mathrm{ind}_a K = p^1(a, K) + 1.$$

(This relation follows at once from the general structure theorems given in my above mentioned paper, Chapter V.)

The theory which I developed in "Kombinatorische Eigenschaften allgemeiner Kurven" gives complete information about the structure of continuous curves of finite connection; it follows from that theory especially, that these curves are identical with the continua K which for every $\epsilon > 0$ can be transformed into a connected one-dimensional complex lying on K by an ϵ-deformation of K in itself.[12] We can therefore consider the problem of characterization of curves which have no points with actually infinite Betti number around them, as being completely solved.

3. Immediate consequences of the above theorem are:

COROLLARY I. *In order that a one-dimensional continuum be a simple closed curve, it is necessary and sufficient that it has the Betti number* 1 *around each of its points.*

COROLLARY II. *In order that a one-dimensional continuum* K *be a simple closed curve, it is necessary and sufficient that its one-dimensional Betti number has the same finite value around all the points of* K. *This value is then equal to* 1.

Using the duality theorem of §2 we have

COROLLARY III. *A one-dimensional continuum* K *lying in the n-dimensional space* R^n *is a simple closed curve if and only if* $q^{n-2}(a, R^n - K)$ *has the same finite value for all points a of* K. *This value is then equal to* 1.

It is known[12a] that a closed set $F \subset R^n$ has the Brouwer-Urysohn-Menger dimensionality r if and only if the following two properties hold:

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[12] See my paper "Über endlich-hoch zusammenhängende stetige Kurven," Fund. Math. 13 (1928), pp. 34–41.

[12a] *Dimensionstheorie*, §4.

1. There exists a point a of F and an $\epsilon > 0$ such that in every neighborhood $U(a, \sigma)$ of a there lies an $(n - r - 1)$-dimensional absolute cycle (with integer coefficients) without common points with F which does not bound any complex (with integer coefficients) lying in $U(a, \epsilon) - F$;

2. If $s < n - r - 1$ then to every $a \subset F$ and $\epsilon > 0$ there belongs a $\sigma > 0$ such that all cycles (with integer coefficients) bound in $U(a, \epsilon) - F$ if they lie in $U(a, \sigma) - F$.

We can now say:

A continuum $K \subset R^n$ is a simple closed curve if and only if the following two properties hold:

1. $q^{n-2}(a, R^n - K) = 1$.

2. *If $s < n - 2$ then to every $a \subset F$ and $\epsilon > 0$ belongs a $\sigma > 0$ such that an s-dimensional cycle bounds in $U(a, \epsilon) - F$ if it lies in $U(a, \sigma) - F$.*

REMARK. The condition $q^{n-2}(a, R^n - K) = 1$ can be again replaced by the weaker condition: $q^{n-2}(a, R^n - K)$ takes in all points a the same finite value.

This gives us a complete characterization of simple closed curves immersed in the n-dimensional space R^n by properties of the complementary domain to these curves in that space.

We have moreover in the special case $n = 2$ the following

THEOREM OF ZARANKIEWICZ. *A plane continuum is a simple closed curve if and only if it locally decomposes the plane in the same finite number of components.*

It can be easily shown by examples that of the two numbers $p^1(a, F)$ and $\operatorname{ind}_a F$ (the last being the Urysohn-Menger index of ramification) the one can be infinite, while the other is finite, even in the case when F is a plane curve.

Let F consist of the point $a = (0; 0)$ and of the points of the circumferences having the intervals $(\frac{1}{2}; 1)$, $(\frac{1}{4}; \frac{1}{2})$, $(\frac{1}{8}; \frac{1}{4})$, $\cdots$, $(\frac{1}{2^n}, \frac{1}{2^{n-1}})$, $\cdots$ as their diameters. Then $p^1(a, F) = \infty$, $\operatorname{ind}_a F = 1$.

As a second example we take the point-set F consisting of the segment $0 \leqq x \leqq 1$, $y = 0$ of the x-axis and of all segments $x = \xi$, $-1 \leqq y \leqq 1$, where ξ belongs to the Cantor perfect set on $[0, 1]$. If a is any point of the form $(\xi; 1)$ or $(\xi; -1)$ then $p^1(a, F) = 0$, $\operatorname{ind}_a F = \mathfrak{C}$; if a is of the form $(\xi; y)$, $-1 < y < 1$, then $p^1(a, F) = 1$, $\operatorname{ind}_a F = \mathfrak{c}$.

By a slight modification of these examples we can exhibit curves F with $p^1(a, F) = \infty$, $\operatorname{ind}_a F = k$ and those with $p^1(a, F) = k$, $\operatorname{ind}_a F = k$, where k is any natural number. It follows, however, from the theorem of the present section that *if* $\operatorname{ind}_a F > 2$ *for all points of the one-dimensional F there must be points a on F with $p^1(a, F) = \infty$.* On the other hand we do not know whether the inequality

$$1 + p^1(a, F) \neq \operatorname{ind}_a F$$

can hold for an individual point a of the curve F in case both $p^1(a, F)$ and $\operatorname{ind}_a F$ are finite (or one of them increasingly-infinite).

One sees in the same way that *there are necessarily points with $p^1(a, F) = \infty$ if for every point a, $p^1(a, F) > 1$.*

We conclude this section with the remark that *on every one-dimensional closed set F the points a with $p^1(a, F) \geqq 1$ form an everywhere dense subset of F*—a result which certainly can be replaced by a stronger one.

5. Accessibility

1. The point a of the closed set $F \subset R^n$ is called an *accessible point* of F if to every component G of $R^n - F$ having the point a on its boundary corresponds a continuum K such that a is the only common point of K and F and $K - a \subset G$. It is easy to see that in this case to every point b of G corresponds a simple Jordan arc $\underline{ba}$ all of whose points, except a, belong to G.

This notion of accessibility fails to be an invariant with respect to topological transformations of F. In fact, let us define F as the sum of the segments S_k, $k = 0, 1, \cdots$, in the xy-plane where S_k, $k > 0$, is the segment $x = 1/k$, $0 \leqq y \leqq 1$ and S_0 is the segment $x = 0$, $0 \leqq y \leqq 1$. Then all points of F are accessible. If on the other hand F' consists of the same segment S_0, the above segments S_{2k} and the segments S'_{2k+1} defined by $x = -1/2k + 1$, $0 \leqq y \leqq 1$, then F and F' are homeomorphic although all the points (x, y), $x = 0$, $0 < y < 1$, are inaccessible points of F'.

One has therefore to replace the above definition of accessibility by another one which is invariant and seems to correspond at least as well as the previous one to the intuitive idea of accessibility. We shall say (with Whyburn)[13] that *$F \subset R^n$ is regularly accessible in its point a if to every $\epsilon > 0$ corresponds a $\sigma > 0$ such that each point of $U(a, \sigma) - F$ can be joined to a by a continuum*

$$K \subset a + [U(a, \epsilon) - F].$$

As a matter of fact it is again easy to see that the continuum K can be always supposed to be a simple Jordan arc.

We do not intend to give here a complete discussion of the relations between the two definitions of accessibility although such discussion presents some interest especially for definite classes of closed sets, for instance for the closed curves in the plane or the closed surfaces in space.[14] However, we wish to emphasize that in any case *the new definition is the stronger one*: if F is regularly accessible in a then a is always an accessible point of F.

2. THEOREM I. *In order that $F \subset R^n$ be regularly accessible in its point a it is necessary and sufficient that to every $\epsilon > 0$ belongs a $\sigma > 0$ such that an arbitrary zero-dimensional cycle z* (with a vanishing sum of coefficients) *on $U(a, \sigma) - F$ bounds on a continuum $K \subset U(a, \epsilon)$ having with F at most the point a in common.*

PROOF. *The condition is necessary.* Every zero-dimensional cycle z^0 with a vanishing sum of coefficients can be represented in the form $\Sigma t^i(b_i - c_i)$, where b and c are zero-dimensional cells i.e. points, we can suppose that $z^0 = b - c$. The point-set F being regularly accessible in a, we can construct the Jordan

[13] G. T. Whyburn, Proceed. Nat. Acad. of Sciences, 13 (1927), 650–657.

[14] As defined in *Gestalt und Lage*, p. 176.

arcs $\overline{ba}$ and $\overline{ac}$ lying in $U(a, \epsilon)$ and having with F only the point a in common. After this we can put $K = \overline{ba} + \overline{ac}$.

The condition is sufficient. We denote by σ_ϵ the number σ corresponding to the ϵ given in our condition. Furthermore let us put $\epsilon_0 = \sigma_\epsilon$ and

$$\epsilon_{k+1} = \text{minimum}\left(\frac{1}{2^{k+1}}, \sigma_{\epsilon k}\right).$$

We take in $U(a, \epsilon_k)$ an arbitrary point c_k, $k = 0, 1, 2, \cdots$, and join for every k the points c_k and c_{k+1} by a continuum $K_{k,k+1}$ lying in $U(a, \epsilon_{k-1}) - F$ and having with F at most the point a in common. In any case $a + \sum_{k=1}^{k=\infty} K_{k,k+1}$ is a continuum joining a with c_0 and having with F only the point a in common.

Theorem I is therefore completely proved.

3. This theorem gives us the means of generalizing the notion of regular accessibility to the case of any number of dimensions. This is why Theorem I is important.

DEFINITION. *The closed set $F \subset R^n$ is called s-accessible in its point a if to every $\epsilon > 0$ belongs a $\sigma > 0$ such that an arbitrary s-dimensional cycle z on $U(a, \sigma) - F$ bounds on a closed set $\Phi \subset U(a, \epsilon)$, having with F at most the point a in common.*

REMARK. From Theorem I the 0-accessibility is the same as the regular accessibility in Whyburn's sense.

As an immediate consequence of this definition and of the known theorems of the general theory of dimensionality[15] (as given by the author) we have:

If the dimension[16] of F is less than r, then F is $(n - r - 1)$-accessible in each of its points.

4. Our main purpose now is to prove the invariance of the property of s-accessibility with respect to topological transformations of F. We begin with the following

LEMMA. *A necessary and sufficient condition in order that F be s-accessible in its point a is that for every ϵ there is a σ_ϵ such that to every s-dimensional cycle $z \subset U(a, \sigma_\epsilon) - F$ and to every $\sigma < \sigma_\epsilon$ there belongs a cycle $z' \subset U(a, \sigma) - F$ which is homologous to z in $U(a, \epsilon) - F$.*

PROOF OF THE LEMMA. *The condition is sufficient.* We have to find a closed set $\Phi \subset U(a, \epsilon)$ having with F at most the point a in common and such that the s-dimensional cycle $z \subset U(a, \sigma_\epsilon) - F$ bounds on Φ. Obviously we can suppose that z does not bound in $U(a, \epsilon) - F$. We put $\epsilon_{-1} = \epsilon$, $\epsilon_0 = \sigma_\epsilon$, $z_0 = z$ and denote by ϵ_{k+1} the smallest of the numbers $\frac{1}{2}^{k+1}$ and $\sigma_{\epsilon k}$; we denote furthermore by z_{k+1} a cycle lying in $U(a, \epsilon_{k+1}) - F$ and homologous to z_k in

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[15] *Dimensionstheorie*, "Allgemeiner Rechtfertigungssatz" on p. 208.

[16] The word "dimension" can be understood here in the Brouwer-Urysohn-Menger sense as well as in the sense "modulo zero" (dimension with respect to the field of rational numbers) given in my above mentioned paper.

$U(a, \epsilon_{k-1}) - F$, $k = 0, 1, 2, \cdots$, by C_{k+1} a complex bounded by $z_k - z_{k+1}$ in $U(a, \epsilon_{k-1})$. Then $a + \sum_{k=0}^{k=\infty} \bar{C}_{k+1}$ is a closed set $\Phi \subset U(a, \epsilon)$ which has only the point a in common with F. The cycle z bounds on Φ.

The condition is necessary. If F is s-accessible in a, and $\Phi \subset a + [U(a, \epsilon) - F]$ is a closed set on which the given cycle $z \subset U(a, \sigma_\epsilon) - F$ bounds, then there exists a sequence

$$C_1, C_2, \cdots, C_k, \cdots$$

of complexes on Φ, where C_k is an ϵ-complex bounded by a subdivision of z and $\lim \epsilon_k = 0$. Then let C_k' be the complex built up of all simplexes of C_k which are within $U(a, \sigma)$, σ being an arbitrary small positive number less than σ_ϵ. If k is sufficiently large, then the simplexes of $C_k - C_k'$ are small enough to have no point in common with F. Therefore

$$z \sim \dot{C}_k' \text{ in } U(a, \epsilon) - F,$$

which completes our proof.

5. DEFINITION. *We say that F has no r-dimensional condensation in its point a if to each $\epsilon > 0$ there belongs a $\sigma_\epsilon > 0$ such that an arbitrary r-dimensional cycle* mod $[F - U(a, \epsilon)]$ *not bounding on F* mod $[F - U(a, \sigma_\epsilon)]$ *does not bound* mod $[F - U(a, \sigma)]$ *for every $\sigma < \sigma_\epsilon$.*

Using our previous terminology we can say: F has no r-dimensional condensation in a, if to each ϵ belongs a σ_ϵ such that every $Z(\epsilon, \sigma_\epsilon)$ is a $Z(\epsilon, \sigma)$ for every $\sigma < \sigma_\epsilon$.

REMARK. An analogous definition holds for every field of coefficients.

THEOREM II. *Let $r \geqq 0$ and $s \geqq 0$ be two integers such that $r + s = n - 1$. A necessary and sufficient condition that F be s-accessible in a is that F has no r-dimensional condensation in that point.*

PROOF. I. *The condition is necessary.* Let F be s-accessible in a. We have to show that there is no r-dimensional condensation in that point. According to the preceding lemma there exists for every ϵ a σ_ϵ such that to every s-dimensional cycle $z \subset U(a, \sigma_\epsilon) - F$ and to every $\sigma < \sigma_\epsilon$ there belongs a cycle $z' \subset U(a, \sigma) - F$ homologous to z on $U(a, \epsilon) - F$. Let us fix this σ_ϵ and choose any $Z(\epsilon, \sigma_\epsilon)$. A $z \subset U(a, \sigma_\epsilon) - F$ linked with the given $Z(\epsilon, \sigma_\epsilon)$ can be found. Then the corresponding z' is also linked with $Z(\epsilon, \sigma_\epsilon)$ in such a way that this $Z(\epsilon, \sigma_\epsilon)$ turns out to be a $Z(\epsilon, \sigma)$.

The condition is sufficient. In other words: If F is not s-accessible in the point a, then a is a point of r-dimensional condensation of F. By the lemma, we have only to prove the following assertion:

ϵ being chosen, let us suppose that to an arbitrary $\delta < \epsilon$ there belongs an s-dimensional cycle z on $U(a, \delta)$ which is not homologous on $U(a, \epsilon) - F$ to any cycle situated in $U(a, 2\sigma) - F$, the positive number $\sigma = \sigma(\delta)$ being suitably chosen. Then there exists a $Z(\epsilon, \delta)$ on $F - \overline{U(a, \sigma/2)}$.

In order to prove this, let us put $F' = F \cdot \overline{U(a, \epsilon) - U(a, \sigma/2)}$. First of all z

does not bound on $U(a, \epsilon) - F'$, because otherwise we would have a $c \subset U(a, \epsilon) - F'$ with $\dot{c} = z$ and if we denote by c_0 the part of c lying on $U(a, \sigma) - F'$

$$c = c_0 + (c - c_0)$$

$$(c - c_0)^{\cdot} = z - z_0 \quad (\text{where } z_0 = \dot{c}_0). \tag{1}$$

As $c - c_0$ is on $(U(a, \epsilon) - U(a, \sigma/2)) - F' \subset U(a, \epsilon) - F$, it would follow from (1) that

$$z \sim z_0 \text{ on } U(a, \epsilon) - F,$$

which contradicts our supposition.

We now put $F'' = F' + S$, where S is the boundary of $U(a, \epsilon)$, thus an $(n-1)$-dimensional sphere. The cycle z cannot bound on $R^n - F''$, because a complex C bounded by z on $R^n - F''$ could be decomposed into the sum of two complexes C_i and C_e the first of them lying in the interior, the second in the exterior of the sphere S. As the boundary z of C lies in the interior of S, we must have $z = \dot{C}_i$ which contradicts what was just proved.

Thus z, as a non-bounding cycle of $R^n - F''$, is linked with an r-dimensional cycle Z of F''. Taking the part of Z within $U(a, \epsilon)$ (i.e. on F') we get a cycle on F' mod $[F - U(a, \epsilon)]$ linked with z and therefore not bounding on F [mod $F - U(a, \delta)$]; this is obviously a $Z(\epsilon, \delta)$ lying on F' and thus on $F - U(a, \sigma/2)$ q.e.d.

6. Let us take

$$\delta < \sigma(\delta)/2,\ \delta_2 < \sigma(\delta_1)/2,\ \cdots,\ \delta_{k+1} < \sigma(\delta_k)/2,\ \cdots .$$

There is on $U(a, \delta_k) - F$ an s-dimensional cycle z_k linked with a certain $Z(\epsilon, \delta_k)$, say Z_k, on $F \cdot \overline{U(a, \epsilon)} - U(a, \delta_{k+1})$. Therefore z_{k+1} cannot link Z_k. It follows from this, that the z_k are independent in $U(a, \epsilon) - F$. Let us consider, in fact, a homology

$$t_1 z_{i_1} + \cdots + t_s z_{i_s} \sim 0 \qquad \text{on } U(a, \epsilon) - F. \tag{2}$$

and denote the linking coefficient $\mathfrak{v}(z_i, Z_j)$ by u_{ij}. Obviously we can suppose that all coefficients in (2) are different from zero. It follows from (2) that

$$t_1 u_{i_1 i_1} + \cdots + t_s u_{i_s i_1} = 0$$

which is impossible because $u_{i_1 i_1}$ is different from 0 while all $u_{i_h i_1} = 0$, $h > 1$. In an analogous way we prove that no homology

$$t_1 Z_{i_1} + \cdots + t_s Z_{i_s} \sim 0 \text{ on } F \bmod [F - U(a, \delta_k)] \tag{3}$$

can hold if all $i_h > k$ and the t_i are different from 0. It would follow in fact from (2) that

$$t_1 u_{i_s i_1} + \cdots + t_s u_{i_s i_s} = 0,$$

which is impossible because $u_{i_s i_h} = 0$ for $h < s$ and $u_{i_s i_s} \neq 0$.

Thus we have proved:

THEOREM III. *If a is a point of r-dimensional condensation of F, then an $\epsilon > 0$ and a sequence of r-dimensional cycles on F mod $[F - U(a, \epsilon)]$, say $Z_1, Z_2, \cdots, Z_k, \cdots$ can be chosen in such a way that*

1. *To each $\sigma > 0$ belongs a k_σ such that the Z_k with $k > k_\sigma$ are independent with respect to homologies on F mod $[F - U(a, \sigma)]$.*

2. *To each k belongs a σ_k such that Z_k is on $F - U(a, \sigma_k)$.*

COROLLARY I. *If $p^r(a, F)$ is finite or increasingly infinite, F has no r-dimensional condensation at the point a.*

COROLLARY II. *A topological polyhedron[17] immersed in a Euclidean space R^n is accessible (with respect to any number of dimensions) in each of its points.*

If F' is a closed subset of the r-dimensional closed set F then obviously $p^r(a, F') \leqq p^r(a, F)$. Therefore:

COROLLARY II. *If the r-dimensional Betti number around every point a of the r-dimensional closed set F is finite or increasingly infinite, then an arbitrary closed subset F' of F is $(n - r - 1)$-accessible in each of its points.*

We have as a special case:

COROLLARY III. *If $F \subset R^n$ is homeomorphic to a subset of an r-dimensional polyhedron, then F is $(n - r - 1)$-accessible in each of its points.*

This contains as a special case a theorem proved by Mazurkiewicz[18] which affirms that a closed set $F \subset R^n$ which is homeomorphic to a subset of R^{n-1} is accessible (in R^n) in each of its points.

7. As the last but, perhaps, most important corollary of Theorem III we mention the following

INVARIANCE THEOREM. *Let F and F' be two homeomorphic closed compact sets situated respectively in R^n and $R^{n'}$. Let F be s-accessible in its point a. If a homeomorphism between F and F' puts the point a' in correspondence with the point a, then F' is $(s + n' - n)$-accessible at the point a'.*

REMARK I. It is easy to see that if F has no r-dimensional condensation at the point a, then to every $\epsilon > 0$ belongs a $\sigma_\epsilon > 0$ such that an arbitrarily chosen $\mathfrak{B}^r_{\epsilon, \sigma_\epsilon}(a, F)$ is at the same time a $\mathfrak{B}^r_{\epsilon, \sigma}(a, F)$ for every $\sigma < \sigma_\epsilon$.

REMARK II. We have pointed out that the notion of r-dimensional condensation can be introduced with respect to any field of coefficients. The same is true for the notion of s-dimensional accessibility; as the definitions remain literally the same it is useless to repeat them here. Now the question arises whether Theorem III with all its corollaries hold for an arbitrary field of coefficients. We cannot give at the present time a definite answer to this question

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[17] A topological polyhedron is by definition a topological image of an Euclidean polyhedron, while the latter is defined as a pointset (of a certain Euclidean space) which can be decomposed into simplexes in such a way that these simplexes form a complex in the ordinary sense of combinatorial Analysis Situs. As a matter of fact, I say "polyhedron" instead of "complex" in all cases in which the corresponding figure is considered as a point-set (and not as a finite or infinite system of simplexes).

[18] Fund. Math. 13 (1929), pp. 146–150.

in its general case, although the affirmative answer seems to us very probable. The proof of Theorem III depends, in fact, on the duality theorem for the corresponding field of coefficients. We have no duality theorem for an absolutely general field, and that is why we have no complete answer to the above question. However modulo m that is in the case of a finite cyclic group chosen as the field of coefficients—all previous arguments hold without any modification; thus Theorem III and its consequences also hold.

8. REMARK III (application to the one-dimensional case). By an argument analogous to that of Section 4, it could be shown without difficulty that a curve, (in the Urysohn-Menger sense) having in no point a one-dimensional condensation, is necessarily locally connected. From the results of my paper "Kombinatorische Eigenschaften allgemeiner Kurven" (referred to in section 4) we have then the following

THEOREM IV. *A closed curve situated in R^n which is $(n-2)$-accessible in each of its points is necessarily a simple closed curve*[19] (i.e. a topological image of a circumference).

We get in this way an exact generalization of the Schoenflies converse of the Jordan-curve-theorem.

6. The Betti groups in and through a point

1. The Betti number around a point gives us only the first and very incomplete information about the local structure of a closed set. For a deeper investigation a localization of the Betti *groups* is needed. A little reflection shows the difficulties which arise when we wish to have Betti groups around a point corresponding to the above Betti number; it even seems to the author that such a definition cannot be found in a natural way for the *general case*, and he hopes that the developments of this paragraph will make the origin of this pessimism clear. In any case, what follows, will lead us to the definition of Betti groups *in* (and *through*) a given point which corresponds to Betti numbers *in* (and *through*) that point, in general different from the Betti numbers *around* the point.

2. We shall now consider cycles and homologies corresponding to an arbitrary abelian group $\mathfrak{J}$ as a *field of coefficients*.[20]

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[19] General closed curves are defined in my paper "Kombinatorische Eigenschaften etc." as well as in *Gestalt und Lage*, p. 176. As a matter of fact, they are one-dimensional continua $F \subset R^n$ characterized by the following two properties:

1. There is a $(n-2)$-dimensional cycle on $R^n - F$ which fails to bound on $R^n - F$;
2. If F' is a true subset of F, then every $(n-2)$-dimensional cycle on $R^n - F'$ bounds on $R^n - F'$.

[20] The notion of a field of coefficients was first introduced in full generality in *Urysohnsche Konstanten*. See also P. Alexandroff, *Einfachste Grundbegriffe der Topologie*, Berlin, Springer, 1932, footnote (23) on page 24, where references will be found to Veblen and Alexander, who first introduced the group of order two as the field of coefficients.

The most important among the different fields of coefficients are the following ones:

1) The field *modulo m*, i.e. the cyclic group of order m (Veblen and Alexander for $m = 2$, Alexander in the general case).

2) The field $\mathfrak{R}$, i.e. the field of all rational numbers (Lefschetz);

3) The field $\mathfrak{R}_1$, i.e. the group of rational numbers reduced modulo 1; the same field can also be defined as the group of rotations of a circle, the angle of rotation being of the form $\frac{p}{q} \cdot 2\pi$, where $p < q$ are arbitrary integers (Pontrjagin and Hopf).

4) The field $\mathfrak{G}$, i.e. the group of all integer numbers (Poincaré).

3. A relative cycle z (belonging to a given field $\mathfrak{F}$) on the closed set F is called a cycle *through* the point a of F, if there exists a neighborhood $U(a)$ of a such that z is a cycle mod $[F - U(a)]$. The r-dimensional cycles (belonging to the field $\mathfrak{F}$) through a obviously form a group, say $\mathfrak{Z}_a^r(F, \mathfrak{F})$. We say now, that the cycle z through a is homologous to zero *in* a if there exists a neighborhood $V(a)$ of a such that z bounds mod $[F - V(a)]$ (with respect to $\mathfrak{F}$); the r-dimensional cycles through a which are homologous to zero in a form a sub-group $\mathfrak{H}_a^r(F, \mathfrak{F})$ of the group $\mathfrak{Z}_a^r(F, \mathfrak{F})$. The factor-group

$$\mathfrak{B}_a^r(F, \mathfrak{F}) = \mathfrak{Z}_a^r(F, \mathfrak{F}) - \mathfrak{H}_a^r(F, \mathfrak{F})$$

will be called the *r-dimensional Betti group of F in the point a with respect to the field of coefficients* $\mathfrak{F}$. If $\mathfrak{F} = \mathfrak{R}$ then the rank of the group $\mathfrak{B}_a^r(F, \mathfrak{R})$ is the r-dimensional Betti number $p_a^r(F)$ of F *in* a.

4. The cycle Z through a is called *declinable* from a if it can be transformed into a cycle which bounds in a by addition of an absolute cycle lying in an arbitrary small neighborhood of a. The cycles through a which are declinable from a obviously form a group, say $\mathfrak{D}_a^r(F, \mathfrak{F})$, which contains the group $\mathfrak{H}_a^r(F, \mathfrak{F})$ as a sub-group:

$$\mathfrak{Z}_a^r(F, \mathfrak{F}) \supset \mathfrak{D}_a^r(F, \mathfrak{F}) \supset \mathfrak{H}_a^r(F, \mathfrak{F}).$$

The factor-group

$$\bar{\mathfrak{B}}_a^r(F, \mathfrak{F}) = \mathfrak{Z}_a^r(F, \mathfrak{F}) - \mathfrak{D}_a^r(F, \mathfrak{F})$$

is called the r-dimensional *Betti group of F through a* with respect to $\mathfrak{F}$. The rank $\bar{p}_a^r(F)$ of the group $\bar{\mathfrak{B}}_a^r(F, \mathfrak{R})$ is called the r-dimensional *Betti number* of F *through a*.

The definition of a declinable cycle can be given in another form. In fact we have:

THEOREM I. *A cycle Z through a is declinable from a if and only if to every neighborhood $U(a)$ there corresponds a decomposition*[21]

$$Z = C' + C''$$

with the following properties:

1) $\bar{C}' \subset U(a)$;

2) $\bar{C}''$ *has a positive distance from* a;

3) $\dot{C}'$ *bounds on* $\bar{U}(a) - a$ (that is, on the closed set $\bar{U}(a) - V(a)$, where $V(a)$ is a sufficiently small neighborhood of a).

PROOF. If such a decomposition exists, and C_1 is the whole complex on $\bar{U}(a) - V(a)$ bounded by $\dot{C}'$, then $C_1 - C'$ is an absolute cycle on $\bar{U}(a)$ and $Z + (C_1 - C') = C'' + C_1$ lies entirely outside a certain neighborhood $W(a)$ and therefore bounds mod $(F - a)$. Thus the condition is sufficient.

Now let Z be a cycle declinable from a. We take an arbitrary neighborhood $U(a)$ of a such that Z, in any case, is a cycle mod $[F - U(a)]$. There is on $U(a)$ an absolute cycle z such that $Z + z$ bounds mod $(F - a)$. That means that by a convenient choice of a whole complex C and of a neighborhood $V(a) \subset U(a)$ we can satisfy the homology-relation

$$\dot{C} = Z + z + Q,$$

Q being a whole complex on $F - V(a)$. Hence it follows that

$$(C \cdot U)^{\cdot} = Z \cdot U + z + Q \cdot U + q,$$

q being a whole complex on $\bar{U} - U$. Now, $(C \cdot U)^{\cdot}$ and z being absolute cycles, we have

$$(Z \cdot U)^{\cdot} = -(Q \cdot U + q)^{\cdot}.$$

As $Q \cdot U + q$ lies outside of $V(a)$, the last relation means that $(Z \cdot U)^{\cdot}$ bounds on $U(a) - a$. Thus, if we put $C' = Z \cdot U(a)$, $C'' = Z - C'$, we get a decomposition of the desired kind.

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[21] An (absolute or relative) cycle on F is, as we know, an infinite sequence of combinatorial cycles (absolute or relative)

$$Z = (z_1, z_2, \cdots, z_k, \cdots)$$

whose vertices belong to F while the diameters of their simplexes converge to zero with $1/k$. We suppose, moreover, that the cycles z_k *converge* in the sense that to every ϵ belongs a n_ϵ such that for $k > n_\epsilon$ the homologies $z_k \underset{\epsilon}{\sim} z_{n_\epsilon}$ hold (i.e. that $z_k - z_{n_\epsilon}$ bounds a complex whose vertices belong to F and whose simplexes are less than ϵ in diameter).

By a decomposition $Z = C' + C''$ is meant a sequence of decompositions $z_1 = c_1' + c_1''$, $z_2 = c_2' + c_2''$, $\cdots$, $z_k = c_k' + c_k''$. A sequence

$$C = (c_1, c_2, \cdots, c_k, \cdots),$$

where the c_k are complexes with vertices on F and with simplexes whose diameters converge to zero with $1/k$, could be called a *whole complex* on F; they are often used in any homology-theory. The closure of the set of all the vertices of the c_k involved is denoted by $\bar{C}$.

COROLLARY. *The cycles Z through a which are declinable from a can be characterized by the following property: For every sufficiently small neighborhood $U(a)$ the absolute cycle $(Z \cdot U(a))^{\cdot}$ bounds on $\overline{U(a)} - a$.*

It just has been proved that this condition is a necessary one. Its sufficiency is rather obvious, because if $\dot{C}_1 = (Z \cdot U(a))^{\cdot}$, $C_1 \subset \overline{U(a)} - a$, then we put $z = C_1 - Z \cdot U(a)$, which gives us a cycle $Z + z$ lying outside of a certain neighborhood of a.

5. If Z is a cycle through a (say mod $[F - U(a)]$) which, being declinable from a, does not bound in a, and therefore does not bound mod $[F - V(a)]$, the cycles z lying in arbitrarily small neighborhoods of a, which can be chosen according to the definition of a declinable cycle, fail to bound on $V(a)$ (because otherwise $Z = (Z + z) - z$ would bound mod $(F - a)$). Thus we have

THEOREM II. *If F is locally connected with respect to the number of dimensions r and to the field of coefficients $\mathfrak{J}$, then the groups $\mathfrak{D}_a^r(F, \mathfrak{J})$ and $\mathfrak{H}_a^r(F, \mathfrak{J})$ and therefore the groups $\bar{B}_a^r(F, \mathfrak{J})$ and $B_a^r(F, \mathfrak{J})$ are identical.*

It is just as easy to prove

THEOREM III. *If F has no r-dimensional condensation in the point a, with respect to $\mathfrak{J}$, the groups $\mathfrak{D}_a^r(F, \mathfrak{J})$ and $\mathfrak{H}_a^r(F, \mathfrak{J})$ and therefore $B_a^r(F, \mathfrak{J})$ and $\bar{B}_a^r(F, \mathfrak{J})$ are identical.*

PROOF. Let Z be a declinable cycle through a, say a cycle mod $[F - U(a, \epsilon)]$. We shall prove that $Z \sim 0$ mod $[F - U(a, \sigma_\epsilon)]$, σ_ϵ being chosen according to the definition of Section 5, No. 5. It is sufficient to show, that for every positive number η,

$$Z \underset{\eta}{\sim} 0 \qquad \text{mod } [F - U(a, \sigma_\epsilon)]\,. \tag{1}$$

Now, the number $\eta < \sigma_\epsilon$ being chosen, we have a cycle $z \subset U(a, \eta)$ such that $Z + z$ bounds mod $[F - V(a)]$, the neighborhood $V(a)$ being chosen in a suitable manner. F having no r-dimensional condensation, we have

$$Z + z \sim 0 \qquad \text{mod } [F - U(a, \sigma_\epsilon)]\,;$$

thus, the diameter of z being less than η,

$$Z \underset{\eta}{\sim} 0 \qquad \text{mod } [F - U(a, \sigma_\epsilon)]\,,$$

q.e.d.

REMARK. As $\mathfrak{H}_a^r(F, \mathfrak{R}) \subset \mathfrak{D}_a^r(F, \mathfrak{R})$, obviously $p_a^r(F) \geqq \bar{p}_a^r(F)$. On the other hand, let $p_a^r(F) \geqq p$, p being a positive integer. Then there exist on F at least p elements $z_1, \cdots, z_p$ of $\mathfrak{Z}_a^r(F, \mathfrak{R})$ whose classes (with respect to $\mathfrak{H}_a^r(F, \mathfrak{R})$) are independent elements of $B_a^r(F, \mathfrak{R})$. We can suppose that the z_i are all cycles mod $[F - U(a, \epsilon)]$, ϵ being conveniently chosen. For every positive number $\delta < \epsilon$, the z_i are independent mod $[F - U(a, \delta)]$. Thus $p^r(a, F) \geqq p$. We have therefore:

THEOREM IV. *For every point a of F, we have*

$$p^r(a, F) \geqq p_a^r(F) \geqq \bar{p}_a^r(F).$$

If F has no r-dimensional condensation in the point a, every $Z(\epsilon, \sigma)$, where $\sigma < \sigma_\epsilon$, is a cycle through a not bounding in a; thus we have:

COROLLARY. *If F has no r-dimensional condensation in its point a, then*

$$p^r(a, F) = p_a^r(F) = \bar{p}_a^r(F).$$

6. We now give some examples which make clear the intuitive meaning of the previous definitions, at least in their elementary cases.

1. Let F consist of the points (x, y), $0 < x \leqq 1/\pi$, $y = \sin 1/x$ of the (x, y)-plane and of the segment $x = 0$, $-1 \leqq y \leqq 1$ of the same plane. We have for the point $a = (0, 1)$

$$p^1(a, F) = \infty, \qquad p_a^1(F) = \bar{p}_a^1(F) = 0.$$

If $a = (0, y)$ with $-1 < y < 1$, then

$$p^1(a, F) = \infty, \qquad p_a^1(F) = p_a^1(F) = 1.$$

2. F consists of the segments $x = 1/n$, $-1 \leqq y \leqq 1$, and of the segment $x = 0$, $-1 \leqq y \leqq 1$. For $a = (0, \pm 1)$

$$p^1(a, F) = p_a^1(F) = \bar{p}_a^1(F) = 0,$$

for $a = (0, y)$, $-1 < y < 1$,

$$p^1(a, F) = \infty, \qquad p_a^1(F) = \bar{p}_a^1(F) = 1,$$

for $a = (1/n, y)$, $-1 < y < 1$,

$$p^1(a, F) = p_a^1(F) = p_a^1(F) = 1.$$

3. F consists of all segments $x = \xi$, $-1 \leqq y \leqq 1$, where ξ is any number belonging to the Cantor perfect set. For every point of F whose ordinate is different from ± 1 we have

$$p^1(a, F) = \infty, \qquad p_a^1(F) = \bar{p}_a^1(F) = 1,$$

while for the points $a = (p, \pm 1)$ of F

$$p^1(a, F) = p_a^1(F) = \bar{p}_a^1(F) = 0.$$

4. F consists of all segments which join the center a of a circumference with the points of a discontinuous perfect set situated on the circumference. Then

$$p^1(a, F) = p_a^1(F) = \bar{p}_a^1(F) = \infty.$$

5. F consists of the circumferences $x^2 + y^2 = 1/n^2$, $n = 1, 2, 3, \cdots$, and of their common center $a = (0, 0)$. Then

$$p^0(a, F) = p^1(a, F) = \infty; \qquad p_a^1(F) = \bar{p}_a^1(F) = 0.$$

6. F consists of the point $a = (0, 0)$ and of the circumferences

$$(x - 3/2^n)^2 + y^2 = 1/4^n, \qquad n = 1, 2, 3, \cdots;$$

$$p^1(a, F) = p_a^1(F) = \infty, \qquad \bar{p}_a^1(F) = 0.$$

7. F consists of the circumferences

$$(x - 1/2^n)^2 + y^2 = 1/4^n$$

(all of which are tangent to each other in their sole common point $a = (0, 0)$). Then

$$p^1(a, F) = p_a^1(F) = \bar{p}_a^1(F) = \infty.$$

REMARK. This example shows that a continuum without condensation in a (with respect to any r and $\mathfrak{Z}$) can fail to be locally connected with respect to a certain r.

8. F consists of n rectilinear segments having a common end-point a (and no other point which belongs to more than one segment). Then

$$p^1(a, F) = p_a^1(F) = \bar{p}_a^1(F) = n - 1.$$

9. F consists of the circumferences $x^2 + y^2 = 1/n^2$, $n = 1, 2, 3, \cdots$, their common center $a = (0, 0)$, and of k segments $\overline{ab_1}, \overline{ab_2}, \cdots, \overline{ab_k}$, lying in the same plane (xy) and having no point in common except a. Then

$$p^1(a, F) = p_a^1(F) = \infty, \qquad \bar{p}_a^1(F) = 0.$$

10. Now the plane (xy) may be considered as the plane (xy) of a three space (xyz). We shall modify the previous example only in one point. Suppose now that the segments $\overline{ab_1}, \overline{ab_2}, \cdots, \overline{ab_k}$, lie in the space and have with the plane (xy) (and each other) only the point a in common. Then

$$p^1(a, F) = p_a^1(F) = \infty; \qquad \bar{p}_a^1(F) = k - 1.$$

Analogous examples can be constructed in more dimensions.

7. If F has no r-dimensional condensation in the point a, the structure of the r-dimensional Betti group of F in that point is particularly simple. Let us denote by $\mathfrak{Z}_{a,\epsilon}^r$ the group of all r-dimensional cycles of F mod $[F - U(a, \epsilon)]$ with respect to $\mathfrak{Z}$. The sub-group of $\mathfrak{Z}_{a,\epsilon}^r$ consisting of those cycles which bound mod $[F - U(a, \sigma_\epsilon)]$ should be denoted by $\mathfrak{H}_{a,\epsilon}^r$ and the factor group $\mathfrak{Z}_{a,\epsilon}^r - \mathfrak{H}_{a,\epsilon}^r$ by $B_{a,\epsilon}^r$. Since to every cycle mod $[F - U(a, \epsilon)]$—say z—corresponds a uniquely defined cycle mod $[F - U(a, \epsilon')]$, for every $\epsilon' < \epsilon$, namely $z \cdot U(a, \epsilon')$, there is a homomorphic mapping of $B_{a,\epsilon}^r$ *into* $B_{a,\epsilon'}^r$. From the condition that F has no r-dimensional condensation in a, it follows, that this mapping is an *isomorphism* between $B_{a,\epsilon}^r$ and a certain sub-group of $B_{a,\epsilon'}^r$. We can therefore consider $B_{a,\epsilon}^r$ as a sub-group of $B_{a,\epsilon'}^r$. Now, if in an ordered set of groups, say $\mathfrak{G}_t$, we always have $\mathfrak{G}_{t'} \subset \mathfrak{G}_{t''}$ when $t' < t''$, then their union $\mathfrak{G}$ (i.e. their sum in set theoretical

sense) is obviously a group: if x' and x'' are two elements of this union, then x' belongs to $\mathfrak{G}_{t'}$ and x'' to $\mathfrak{G}_{t''}$; e.g. let $t'' > t'$; as x' and x'' both belong to $\mathfrak{G}_{t''}$, the element $x' + x''$ of $\mathfrak{G}$ is well defined. In this way the union of all the groups $B^r_{a,\epsilon}$ is defined as a group, and it is very easy to see that this group is isomorphic to the group $B^r_a(F, \mathfrak{J})$.

8. We shall now introduce the notion of the *Betti groups* of $R^n - F$ *at the point* a of F, this notion being useful if F has no r-dimensional condensation in that point.

We understand by an *s-dimensional cycle* of $R^n - F$ at the point $a \subset F$ a sequence

$$z^r = (z_1, z_2, \cdots, z_k, \cdots)$$

of s-dimensional absolute cycles on $R^n - F$, such that to every $\sigma > 0$ there corresponds a $k(\sigma)$ with the property that all z_k with $k > k(\sigma)$ are on $U(a, \sigma) - F$ and are homologous to each other on $U(a, \sigma) - F$.

The cycles z^s obviously form a group $\mathfrak{Z}^s_a(R^n - F)$, when we put

$$z^s + z'^s = (z_1 + z'_1, z_2 + z'_2, \cdots, z_k + z'_k, \cdots)$$

$$-z^s = (-z_1, -z_2, \cdots, -z_k, \cdots).$$

We say, furthermore, that z^s bounds (at the point a) if all z_k, with k sufficiently large, bound on $U(a, \epsilon) - F$, for every $\epsilon > 0$.

The bounding z^s obviously forms a sub-group $\mathfrak{H}^s_a(R^n - F, \mathfrak{J})$ of the group $\mathfrak{Z}^s_a(R^n - F)$. The factor group

$$B^s_a(R^n - F, \mathfrak{J}) = \mathfrak{Z}^s_a(R^n - F, \mathfrak{J}) - \mathfrak{H}^s_a(R^n - F, \mathfrak{J})$$

is by definition the s-dimensional Betti-group of $R^n - F$ at the point a. The definition holds for every field of coefficients $\mathfrak{J}$; for $\mathfrak{J} = \mathfrak{R}$ we get the Betti number $p^s_a(R^n - F)$ as the rank of $B^s_a(R^n - F, \mathfrak{R})$.

If F has no r-dimensional condensation in a and $\mathfrak{J}$ is one of the coefficient-fields $\mathfrak{R}$ or $\mathfrak{G}_m$ then to every $Z(\epsilon, \sigma)$, $\sigma < \sigma_\epsilon$, belongs a $Z(\sigma, \epsilon)$ linked with this $Z(\epsilon, \sigma)$. As this $z(\sigma, \epsilon)$ is homologous to a $z(\sigma', \epsilon)$ for every $\sigma' < \sigma$, we have a z^s linked with the given $Z(\epsilon, \sigma)$. On the other hand, if a non-bounding $z^s = (z_1, z_2, \cdots, z_k, \cdots)$ is given, there exists an ϵ such that no one of the z_k corresponding to a sufficiently large k bounds on $U(a, \epsilon) - F$. Taking a z_k on $U(a, \sigma_\epsilon) - F$, we find a corresponding $Z(\epsilon, \sigma_\epsilon)$ linked with that z_k and therefore with all but a finite number of the z_k. We say then that $Z(\epsilon, \sigma_\epsilon)$, which is a cycle through a, links the given z^s. Thus we get:

SECOND DUALITY THEOREM. *Let F have no r-dimensional condensation in its point a. To each r-dimensional cycle on F through a which does not bound in a, corresponds a linking cycle of $R^n - F$ at a. To each $(n - r - 1)$-dimensional*

cycle of $R^n - F$ at a not bounding at a belongs a linking cycle on F through a. That is

$$p_a^{n-r-1}(R^n - F) = p_a^r(F)$$

(and therefore $p^r(a, F) = p_a^r(F) = p_a^{n-r-1}(R^n - F) = p^{n-r-1}(a, R^n - F)$).

This theorem holds for $\mathfrak{R}$ and $\mathfrak{G}_m$ as fields of coefficients.

REMARK. As a matter of fact the recent results of Pontrjagin concerning the duality between the group of the integer numbers and the group of the real numbers reduced modulo 1 hold in our local case also.

9. The methods used in this paragraph can be applied to other questions. I shall give here two examples.

1) We consider the *absolute* cycles on F; they form a group which we call $\mathfrak{Z}^r(F, \mathfrak{J})$; in this group are contained the sub-group $\mathfrak{D}^r_{a,F}(F, \mathfrak{J})$ of the cycles which are declinable from a as well as the sub-group $\mathfrak{H}^r_{a,F}(F, \mathfrak{J})$ of cycles bounding in a. Thus we get the groups

$$B^r_{a,F}(F, \mathfrak{J}) = \mathfrak{Z}^r(F, \mathfrak{J}) - \mathfrak{H}^r_{a,F}(F, \mathfrak{J}),$$

$$\bar{B}^r_{a,F}(F, \mathfrak{J}) = \mathfrak{Z}^r(F, \mathfrak{J}) - \mathfrak{D}^r_{a,F}(F, \mathfrak{J})$$

If F has no r-dimensional condensation in a the last two groups are identical. They lead to the corresponding Betti numbers. If $r = n - 1$, $F \subset R^n$, the Betti number $p^{n-1}_{a,F}$ gives us *the number of components of $R^n - F$ containing the point a on their boundary.* Thus *this number turns out to be an invariant* (with respect to topological transformations of F) *at least in case F has no $(n-1)$-dimensional condensation at the point a.*

2) The second application concerns the Brouwer—"*Unbewalltheit*" ("uniform local connectedness" of a domain in R_n). Let us consider an $F \subset R^n$ having the following property at the point a: to each ϵ belongs a σ such that every r-dimensional cycle z^r on F mod $[F - U(a, \epsilon)]$ which does not bound mod $[F - U(a, \sigma)]$ satisfies for every $\eta > 0$ the homology

$$z \underset{\eta}{\sim} Z, \text{ mod } [F - U(a, \epsilon)],$$

Z being an absolute cycle on F, depending on η and (if η is sufficiently small) obviously not bounding on F. We say, if this property holds, that *every r-dimensional cycle on F around a allows a continuation.*

Now let us suppose that $\mathfrak{J} = \mathfrak{R}$ (or $\mathfrak{J} = \mathfrak{G}_m$). Then it follows from the duality theorem of Section 2 that the continuation property we just introduced is equivalent to the following one concerning the complementary space:

To every ϵ belongs a σ such that every $(n - r - 1)$-dimensional cycle lying in $U(a, \sigma) - F$ does not bound on $R^n - F$ if it fails to bound on $U(a, \epsilon) - F$. For a $z(\sigma, \epsilon)$ necessarily links a $Z(\epsilon, \sigma)$, a non-bounding (absolute) cycle on F, if the continuation property holds. Let us assume on the other hand that no $z(\sigma, \epsilon)$ bounds on $R^n - F$; then every $z(\sigma, \epsilon)$ would link an absolute cycle

$Z(\epsilon, \sigma)$ so that we could find a $\mathfrak{B}^r_{\epsilon,\sigma}$ consisting of absolute cycles only. This means (by the definition of $\mathfrak{B}^r_{\epsilon,\sigma}$) that whatever η may be chosen, each $Z(\epsilon, \sigma)$, say z, satisfies an homology

$$z \underset{\eta}{\sim} Z, \text{ mod } [F - U(a, \sigma)]$$

where Z is a finite linear combination of the elements of $\mathfrak{B}^r_{\epsilon,\sigma}$, and hence an absolute cycle.

This gives us, for $r = n - 1$: The continuation property (applied to the $(n - 1)$-dimensional cycles) is equivalent to the following property of the complementary space: to each ϵ corresponds a σ such that two points of R^n whose distance from a is less than σ and which belong to the same component of $R^n - F$, can always be joined within $U(a, \epsilon)$ by a polygonal line not meeting F. If F is the common boundary of components of $R^n - F$, this property is nothing else than the property of uniform local connectedness of all these components. Thus we have the following result:

THEOREM V. *Let F be the common boundary of all domains in which the space is decomposed by F; the property of uniform local connectedness of these domains is invariant with respect to topological transformations of F. This property is equivalent indeed to the property that each $(n - 1)$-dimensional cycle on F around any point of F allows a continuation.*

7. Applications to the theory of dimensionality

1. The point a is called a *nuclear point* of F with respect to $\mathfrak{Z}$, if there exists a subset F' of F such that the group $\bar{B}^r_a(F', \mathfrak{Z})$ is different from zero, r being the dimensionality[22] of F with respect to $\mathfrak{Z}$.

As the Brouwer-Urysohn-Menger dimensionality of F is identical with its dimensionality with respect to $\mathfrak{R}_1$ (where $\mathfrak{R}_1$ is the additive group of all rational numbers reduced *modulo one*)[22a] the nuclear points corresponding to the Brouwer dimensionality are those with respect to $\mathfrak{R}_1$ as field of coefficients.

THEOREM I. *Every closed point-set contains nuclear points with respect to any field of coefficients. If $\Delta\mathfrak{Z}(F) = r$ and $N(F)$ is the set of all nuclear points of F, then $N(F)$ necessarily contains r-dimensional[23] closed sets while there is no r-dimensional closed subset of $F - N(F)$.*

A nuclear point of F being *a fortiori* a nuclear point of every $F_1 \supset F$, our theorem follows at once from

THEOREM I′. *Let $\Delta\mathfrak{Z}(F) = r$. Then F contains an r-dimensional F' such that in all points of F', except those of a nowhere dense subset F'' of F', the group $\bar{B}^r_a(F', \mathfrak{Z})$ is different from zero. If $\mathfrak{Z} = \mathfrak{R}_1$ (that is, in the case of the Brouwer dimensionality), F'' can be supposed to be at most $(r - 1)$-dimensional.*

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[22] See "Dimensionstheorie" and "Urysohnsche Konstanten."

[22a] As proved by Hopf, Pontrjagin and the author in a paper to appear in "Compositio Mathematica."

[23] r-dimensional of course, with respect to $\mathfrak{Z}$.

PROOF OF THE THEOREM I'. As $\Delta\mathfrak{Z}(F) = r$, there is on F an $(r-1)$-dimensional absolute cycle

$$Z^{r-1} = (z_1, z_2, \cdots, z_k, \cdots)$$

with respect to $\mathfrak{Z}$, which bounds on F but at the same time does not bound on a certain subset Φ''; for this set Φ'' we can take the minimal closed subset[24] of F containing the cycle Z^{r-1}.

Now let Φ' be a minimal subset of F containing Φ'' and such that Z^{r-1} bounds on it.[25] I say that the group $\bar{B}_a^r(\Phi', \mathfrak{Z})$ is different from zero at every point a of $\Phi' - \Phi''$. In order to prove this we consider the following cycle mod Φ'' on Φ',

$$Z^r = (c_1, c_2, \cdots, c_k, \cdots),$$

where c_k is an ϵ_k-complex ($\lim \epsilon_k = 0$) on Φ' bounded by z_k^{r-1}. We have only to show that if $U(a)$ is a sufficiently small neighborhood of a (with respect to Φ'), no decomposition $Z^r = C' + C''$ exists with $\bar{C}' \subset U(a)$, $a \subset \Phi' - \bar{C}''$, $\dot{C}' \sim 0$ on $\overline{U(a)} - a$. Now, suppose that such a decomposition exists for a suitably chosen $U(a) \subset \Phi' - \Phi''$. We can then find a neighborhood $V(a) \subset U(a)$, $V(a) \subset \Phi' - \bar{C}''$, and a $C_0 \subset \overline{U(a)} - V(a)$ with $\dot{C}_0 = \dot{C}'$. Then

$$\dot{C}'' = \dot{Z}^r - \dot{C}' = Z^{r-1} - \dot{C}',$$

$$(C_0 + C'')^{\cdot} = Z^{r-1}.$$

As $\bar{C}_0$ and $\bar{C}''$ are both on $\Phi' - V(a)$, the cycle Z^{r-1} bounds on $F' - V(a)$ which contradicts the definition of F'. We now put $F' = \overline{\Phi' - \Phi''}$. The set $F'' = F' \cdot \Phi''$ is obviously nowhere dense on F'. As F' and Φ' are locally identical in the neighborhood of any point $a \subset \Phi' - \Phi''$, the group $\bar{B}_a^r(F', \mathfrak{Z})$ is different from zero. In the case of Brouwer dimensionality, Φ'', and therefore F'', can be supposed $(r-1)$-dimensional.[26]

As an easy corollary to this theorem we have the following *local characterization* of dimensionality:

THEOREM II. *The Brouwer dimensionality of F is the largest number r having the following property: There exists on F a point a such that the r-dimensional Betti group of F in a with respect to $\mathfrak{R}_1$ is different from zero.*

PROOF. We have to show that if $\dim F = r$, then in no point a of F the group $B_a^s(F, \mathfrak{R}_1)$, $s > r$ is different from zero, while in some points a of F the group $B_a^r(F, \mathfrak{R}_1)$ does not vanish.

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[24] See *Gestalt und Lage*, p. 181–182.

[25] "Minimaler Homologieträger"—see *Gestalt und Lage*, pp. 181–182. Such an F' always exists because if Z^{r-1} bounds on $F_1, F_2, \cdots, F_k, \cdots, F_k \supset F_{k+1}$, then Z^{r-1} bounds on the intersection F_ω of all these F_k.

[26] As follows from *Dimensionstheorie*, Nr. 66, p. 213.

The first assertion follows from the fact that an s-dimensional cycle absolute or relative, belonging to any field of coefficients bounds on F, if the Brouwer dimensionality of F is less than s.[27]

In order to show that in a certain point a of F the group $B_a^r(F, R_1)$ does not vanish, we first remember that there exists, according to theorem I′ a certain r-dimensional cycle through $a \subset F' \subset F$, say Z^r, which is not declinable from a on F' and therefore does not bound on F' in a. This means that there exists a neighborhood $U(a)$ (with respect to F) such that Z^r is a cycle on F' mod $[F' - U(a)]$, and thus *a fortiori* a cycle on F mod $[F - U(a)]$ which does not bound on $F' - V(a)$ for any $V(a) \subset U(a)$. I say that Z^r fails to bound on F mod $[F - V(a)]$. Let us suppose, in fact, that there exists a C^{r+1} bounded by Z^r on F mod $[F - V(a)]$. Let Z' be the complete boundary of C^{r+1}; we have in Z' an absolute r-dimensional cycle which certainly does not bound on $F' + (F - U(a))$, because otherwise Z^r would bound on F' mod $[F' - V(a)]$. As Z', being thus an essential ("wesentlich") r-dimensional cycle, bounds by definition on F, the dimensionality of F must be at least equal to $r + 1$. This contradiction shows that Z^r really does not bound on F in a, so that the r-dimensional Betti group of F in a (with respect to $\mathfrak{R}_1$) turns out to be different from zero.

Moreover, from the remark made at the beginning of the proof we have:

THEOREM II$_\mathfrak{J}$. *The Brouwer dimensionality of F is the largest number r having the following property. There is a point a on F and an Abelian group $\mathfrak{J}$ such that the r-dimensional Betti group of F in a with respect to $\mathfrak{J}$ as the field of coefficients is different from zero.*

In the same way as we proved Theorem II, we get the characterization of dimensionality modulo m:

THEOREM II′. *Let $\mathfrak{J}$ be the field of all rational numbers, or the field modulo m. The dimensionality of F with respect to $\mathfrak{J}$ is the largest number r having the property that there exists a point a of F in which the group $B_a^r(F, \mathfrak{J})$ is different from zero.*

REMARK. It follows from Theorem I that the Brouwer dimensionality of F is the greatest number r such that there exists a subset F' of F, and a point a of F' in which the group $\bar{B}_a^r(F_1, \mathfrak{R}_1)$ or the group $B_a^r(F, \mathfrak{J})$ for *any* $\mathfrak{J}$) is different from zero. An analogous modification also holds for Theorem II′. We finally point out, that *the dimensionality of F with respect to $\mathfrak{R}$ is the largest number r having the property, that there exists a point m of F such that F has a non-vanishing r-dimensional Betti number around that point.* Once more we can replace here the Betti number *around* a point by the Betti number *in* a point.

Unfortunately we cannot, at the present time, prove Theorem II′ for an *arbitrary* field of coefficients (although there is every reason to suppose that it is true without any exception). This is because we use duality relations in order to prove that on an r-dimensional set every $(r + 1)$-dimensional relative cycle

[27] This is an immediate consequence of the deformation theorem proved in *Gestalt und Lage*, p. 120.

must bound ("Dimensionstheorie" No. 48, p. 199), and the generalization of these relations to the case of an arbitrary field of coefficients is still an unsolved problem.

2. THEOREM III. *Let F be an n-dimensional point-set in R^n. The nuclear points of F are identical with the interior points of F with respect to R^n.*

This theorem belongs entirely to the well known category of elementary "Invariance-theorems." It is sufficient to sketch the proof.

First let a be an interior point of F, T^n an n-dimensional simplex lying in F and containing a in its interior. By the well known method of ϵ-displacements we can reduce every n-dimensional cycle on F through a and every homology in a of such a cycle to a combinatorial cycle and a combinatorial homology on a certain subdivision of T^n. The latter cycles and homologies are taken mod $(T^n - t^n)$, where t^n is a simplex similar to T^n, containing a in its interior and lying entirely in the interior of T^n. It follows from this at once that:

1. T^n (considered as a cycle through a, mod $(F - T^n)$) does not bound in a;
2. Every n-cycle through a is homologous mod $(F - t^n)$ to a cycle of the form $u\,t^n$, u being an element of the corresponding field of coefficients $\mathfrak{F}$;
3. There is no n-dimensional condensation in a;
4. The n-dimensional Betti groups of F in and through a are both isomorphic with the group $\mathfrak{F}$. The point a is therefore a nuclear point of F.

Now, let a be a nuclear point of F. Then there exists an ϵ and an n-cycle z^n on F mod $[F - U(a, \epsilon)]$ which does not bound mod $[F - U(a, \epsilon)]$. By an arbitrary small displacement we then get an n-dimensional polyhedral cycle z' mod $[R^n - U(a, \epsilon)]$ which in any case is different from 0 and therefore fills up the whole $U(a, \epsilon)$; thus F is everywhere dense on $U(a, \epsilon)$ and being closed contains $U(a, \epsilon)$; a is therefore an interior point of F, q.e.d.

3. There are in the literature different attempts to give an adequate definition for the intuitive notion of "nuclear" and "shell" points of a closed point set.[28]

As a matter of fact, the definition I gave in my "Dimensionstheorie" (loc. cit.) and the present one both lead to the same properties expressed by Theorems I and III. These same properties hold also for the definition of Borsuk. On the other hand, the theorems that Borsuk proved for his definition do not differ from those of the present paper.[29] It would be very interesting to know *whether the definition of nuclear points given by Borsuk is equivalent to that of the present paper.*

[28] For instance: Baer and Levi, Ränder topologischer Räume, Berichte d. Sächsisch. Akad. d. Wiss. 82 (1930), pp. 171–189;

Alexandroff, *Gestalt u. Lage*, p. 185 and *Dimensionstheorie*, Nr. 68, p. 214;

Borsuk, Ergebnisse eines mathematischen Kolloquiums, 1932, 49th Colloquium.

[29] Borsuk proves moreover that for a locally connected curve (in the sense of Urysohn and Menger) the nuclear points (in his sense) are identical with the points whose Urysohn-Menger ramification index is greater than 1. It is an easy matter to show that the same property also holds for our definition of nuclear points.

I see a considerable advantage in my present definition of nuclear points (when compared with my old definition as well as with that of Borsuk) in that it deals with the notion of local Betti group only. Thus we see that various structure characters of closed sets: dimensionality, nuclear and "shell"-points, condensation, ramification, accessibility, have their origin in the same few elementary notions. This enables us to make a much more systematic study of the topology of closed sets than is possible by set-theoretical methods alone. The fact that cycles and homologies form the only apparatus of such a study, guarantees its elementary character.

The definition[30] given in my "Dimensionstheorie" rests also upon cycles and homologies but it is more or less a definition *ad hoc*, which does not make clear the connexions between the nuclear points and various fundamental notions of the modern topology of closed sets. This is why I abandoned the older definition for the present one. The latter allows us, by the way, to introduce the notion of a "nuclear set" in the same manner as in *Dimensionstheorie*, and to prove a theorem completely analogous to, but stronger than the "4th Hauptsatz" of *Dimensionstheorie*.

4. The nuclear points as defined in No. 1 of this section could be called nuclear points in the *strong sense*. We get nuclear points in the *weak sense* if we replace in the definition of No. 1 the group $\bar{B}_a^r(F, \mathfrak{Z})$ by the group $B_a^r(F, \mathfrak{Z})$. As a nuclear point in the strong sense is a nuclear point in the weak sense, Theorem I is *a fortiori* true for the nuclear points in the weak sense. Theorem II holds true also for the nuclear points in the weak sense. Two elementary examples make the situation clear:

1. F is the curve y-sin $1/x$ with the segment $(-1, 1)$ of the y-axis (example VI of Section 1); a is the point $(0; 1)$. Then a is a nuclear point in the sense of *Dimensionstheorie*; however a is not a nuclear point even in the weak sense.

2. F is the point-set of example 2 of Section 3, No. 3. The point a of this set is a nuclear point in the weak sense; it is neither a nuclear point in the strong sense, nor a nuclear point in the sense of *Dimensionstheorie*.

To conclude these remarks, we mention a numerical invariant which seems worth further investigation: It is the upper bound of all numbers $\bar{p}_a^r(F')$, F' being any r-dimensional subset of the r-dimensional F; this number (finite or infinite) could be called the *order of connectivity* of F at the point a.

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[30] I say there that a is a direct nuclear point of the r-dimensional set F if the boundary B of every sufficiently small neighborhood $U(a)$ of a (with respect to F) contains $(r-1)$-dimensional cycles which bound on $U(a) + B$ but fail to bound on B. A point a is called a nuclear point of F if there is an r-dimensional sub-set F' of F, to which a belongs as a direct nuclear point.

It would be easy to show that every nuclear point in the sense of the present paper is a nuclear point in the sense of "Dimensionstheorie," while the converse is in general not true.

8. Some remarks and unsolved problems

1. Let dim $F = r$. We suppose that F has no r'-dimensional condensation, for any $r' \leqq r$ and let the field of coefficients be $\mathfrak{J}$. Under such conditions we say that F has in the point a no condensation at all. We suppose that this is really the case with the point a. We then say, furthermore, that F *has no* $(r - 1)$*-dimensional torsion* in the point a with respect to $\mathfrak{J}$, if the group $B_a^r(F, \mathfrak{J})$, (which is in our case isomorphic with the group $\bar{B}_a^r(F, \mathfrak{J})$), is the direct sum of $p^r(a, F)$ sub-groups, each of them isomorphic with the group $\mathfrak{J}$. Points a of F in which there is neither condensation nor torsion for any $\mathfrak{J}$ and r, are called *regular points* of F. The homology characters of F in a regular point are entirely governed by the Betti numbers of F *in* (or, what is in this case the same, *around*) that point. A regular point is a *nuclear point*, if the r-dimensional Betti number of F is positive in this point, and is a shell point otherwise. The regular point a is a point of *r'-dimensional ramification*, $0 < r' < r$, if the r'-dimensional Betti number of F is positive in a; the point a is a point of *r-dimensional ramification*, if $p^r(a, F) > 1$ (we remember that $r =$ dim F). If F has no ramification in the point a for any $r' > 0$, the point a is an *ordinary* point.

A closed bounded point-set F (in other words: a compact metric space of finite dimensionality), all points of which are regular points, is called a *regular*[31] point-set. A regular set, whose points are all ordinary points, is what I call a *compact* (in general, bounded) *topological manifold*.[32] If all points of a topological manifold are nuclear points, then the manifold is called *closed*. *Thus an r-dimensional closed topological manifold is a point-set F having neither condensation nor torsion in any point and whose local Betti numbers are those of an interior point of an r-simplex.*

2. An outstanding problem which seems to me of great importance is to show that the Lefschetz intersection theory (including the Duality-theorem) holds for topological manifolds in the sense here given (*Problem I*). I do not think that this problem contains any difficulty of principle: I suppose, indeed, that the proof will go roughly speaking along the same lines as Lefschetz' proof for his manifolds. *Be that as it may, there is no doubt that the solution of the problem of general topological manifolds was given by Lefschetz (loc. cit.).*[32a] If I suggest here a new definition, it is only because I think that it gives us a way to include the notion of manifold in the system of elementary notions, as con-

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[31] This seems to be a natural generalization to the r-dimensional case of the *curves of finite connexion*, which I studied in the last chapter of my "Kombinatorische Eigenschaften allgemeiner Kurven," Math. Ann. 96 (1926), pp. 512–555.

[32] I suggest calling the topological manifolds in the sense of Hopf (Math. Ann. 100 (1928), pp. 579–608) *locally-Euclidian* topological manifolds (or simply *Hopf-manifolds*), and the generalized manifolds introduced by Lefschetz—the *Lefschetz manifolds* (Amer. Journ. of Math. 45 (1933)), 469–504.

[32a] See also a Memoir of Čech (These Annals (2), 34 (1933)).

densation, Betti groups in a point, and so on, which seem to present a general topological interest independent of any definition of a manifold. As a matter of fact, it would be very interesting to know, whether the Lefschetz definition is equivalent to the one just given here (*Problem II*).

Another unsolved problem is the following one:

PROBLEM III. *Does there exist, for each point a of an r-dimensional closed topological manifold, a neighborhood $U(a)$ such that the Betti groups of $\overline{U(a)}$* mod *$[\bar{U}(a) - U(a)]$ are the same as the corresponding groups of an r-dimensional simplex modulo its boundary?*

3. I wish to point out that the definition of manifolds given above (as well as the whole theory of local invariants given in the present paper) can be generalized without any difficulty to the case of locally compact spaces, which would give us a definition of open manifolds. On the other hand, one can generalize this theory to the case of a bicompact space with the Abzählbarkeitsaxiom I of Hausdorff (i.e. to spaces in which every point has a countable system of neighborhoods). Concerning the manifolds, it would seem advisable, however, to show, before undertaking a generalization to the case of arbitrary bicompact spaces, that there really exist non-metrizable bicompact spaces satisfying one or the other manifold definition; there is otherwise a danger of building up an extensive theory without a real object! To give an example: it follows from the recent manifold definition by Čech, that a manifold in his sense is a locally bicompact space with the Abzählbarkeitsaxiom I of Hausdorff, because this axiom follows from the Axiom A_2 of Čech,[33] as I proved in my paper "Über die Metrisation der im Kleinen kompakten topologischen Räume" (Math. Ann. 92 (1924), p. 296, Satz I; see also Alexandroff and Urysohn, Mémoire sur les espaces topologiques compacts, Verhand. Akad. Amsterdam, Deel XIV (1929), No. 1, p. 65, Theorem III). Of course, a bicompact space with the Abzählbarkeitsaxiom I may not be metrizable (see the just mentioned "Mémoire," p. 76, example A_7), but I do not know, whether under the other conditions of Mr. Čech the second Abzählbarkeitsaxiom of Hausdorff (the "separability" and thus the possibility to introduce a metric) follows from the first. I do not even know the answer on the following questions:

PROBLEM IV. *Does there exist a connected and locally connected in all dimensions non-metrizable bicompact space, which satisfies the first "Abzählbarkeitsaxiom" of Hausdorff?*

PROBLEM IV′. *Does there exist a* (closed) *manifold in the sense of Čech which is not metrizable; does there exist a manifold in the sense of Čech which is not a Lefschetz manifold?*

4. It is easy to see that the r-dimensional Betti group of a polyhedron in a point a is isomorphic with the r-dimensional Betti group of the star of the point

[33] These Annals (2) 34 (1933), p. 625.

a (in any simplicial subdivision of the polyhedron) modulo the boundary of the star, thus to the $(r-1)$-dimensional (absolute) Betti group of that boundary. *A polyhedron is therefore a regular point-set if and only if the torsion coefficients of the boundaries of its stars* (in any simplicial subdivision) *vanish.* It follows from this, that *a polyhedron, which is a topological manifold in our sense is necessarily a manifold in the sense of Alexander-van Kampen-Pontrjagin.*[34] As the converse is obviously also true (every h-manifold being a topological manifold in our sense) we see that the Alexander-van Kampen-Pontrjagin manifolds are nothing else but *polyhedral topological manifolds.*

5. There are many problems concerning regular point sets and connected notions. For instance

PROBLEM V. *What is the distribution of points of different indices*[35] *on a regular point-set? Is a regular point-set all of whose points have the same index necessarily a closed manifold? Can a point-set without condensation contain points with an actually infinite Betti number around them?*

PROBLEM VI. *What relations are there between the absence of condensation* (in all dimensions) *and the local connectedness* (also in all dimensions)*?*

PROBLEM VII. *What kind of point-sets form the various types of points on F* (the condensation points, the points with torsion, the nuclear points, the points with a given index on a regular point-set and so on)? *What is the Baire-Lebesgue class of these sets, their dimensionality and so on?*

PROBLEM VIII. *To prove Theorem II′ of section 7 for every field of coefficients.* This problem seems to involve an exposition of the Theory of dimensionality (with respect to any field of coefficients) in which intrinsic properties of F would be proved by intrinsic means (that is without use of the properties of the complementary space).

PROBLEM IX. *By what kind of approximating complexes* ("projective spectrum" in the sense of *Gestalt und Lage,* p. 107) *are the topological manifolds characterized? The same problem for regular point-sets.*

6. Theorem $\text{II}_{\mathfrak{Z}}$ of section 7 states that if there exists a field of coefficients $\mathfrak{Z}$ and a point a such that $B_a^r(F, \mathfrak{Z})$ is different from zero, then we can always take $\mathfrak{R}_1$ for such a field of coefficients. The question now arises: *is there a field of coefficients $\mathfrak{Z}_0$, such that from the existence of*—say *r-dimensional-condensation of F in a given point a with respect to any field of coefficients, follows the analogous phenomenon with respect to $\mathfrak{Z}_0$* (at the point a or at an other point)? This question is intimately connected with the following one:

PROBLEM X. *Does there exist a field of coefficients $\mathfrak{Z}_0$ having the following property: for any closed point set F and the Abelian group $\mathfrak{Z}$, the Betti group $B^r(F, \mathfrak{Z})$ (of F with respect to $\mathfrak{Z}$) can be expressed by means of $B^r(F, \mathfrak{Z}_0)$ (and the group $\mathfrak{Z}$ itself)?*

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[34] See for instance Pontrjagin, Math. Ann. 105 (1931), p. 181; Pontrjagin calls them h-manifolds (which means "Homology"-manifolds).

[35] The index of a regular point is the system of all Betti numbers around it.

If we confine ourselves to the case of polyhedra, then (as follows from unpublished results of Hopf), the group of integer numbers as well as the group $\mathfrak{R}_1$ can be taken for the "universal" field of coefficients $\mathfrak{Z}_0$.

An analogous question can be proposed for the local Betti groups (in and through a point).

There are some reasons to suppose that the Pontrjagin field of coefficients, namely the group of all real numbers reduced modulo 1 is such a universal field (for all closed sets), but this hypothesis is still unproved.

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CONDITIONS FOR FACTORIZATION IN A SET CLOSED UNDER A SINGLE OPERATION

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1°. Consider a system S consisting of a set of elements $a, b, c, \cdots$ over which an equality relation and a binary operation multiplication have been defined satisfying the following four conditions.

P 1 *To every pair of elements a, b of S there corresponds an element c of S unique to within equal elements.* We write $c = ab$.

P 2 *If $a = a'$ and $b = b'$ then $ab = ab' = a'b = a'b'$.*

P 3 *$a(bc) = (ab)c$ for all a, b, c in S.*

P 4 *$ab = ba$ for all a, b in S.*

Recently A. H. Clifford[1] and others have considered the following problem: to determine what additional conditions it is necessary to impose on S in order that each integral element in it may be uniquely resolved into a product of powers of irreducible elements (up to unit factors) as in the case when S is the set of positive integers and the operation ordinary multiplication.[2]

I propose to show here that we can discard the requirement of *unicity* in the resolution into prime factors and still retain many of the essential features of common arithmetic. The set of conditions which I shall develop will assign to each integral element a certain canonical decomposition into prime factors to which it is equivalent. If we know this canonical decomposition, we know all divisors of the element and the canonical decomposition of each divisor. Any two elements of the system will have a least common multiple, and if they have any common divisors, a greatest common divisor. On the other hand, we cannot obtain the canonical decomposition of a product from a knowledge of the canonical decompositions of its factors, nor need irreducible elements be primes.[3]

We shall show that our system includes as special instances the arithmetics previously discussed by J. Koenig,[4] Clifford,[1] Fritz Klein[5] and others.

2°. We must first lay down a few definitions. a is said to divide b either if

[1] Bulletin Am. Math. Soc. 40 (1934), pp. 326–330. We shall refer to this paper as Clifford.

[2] We refer to this case hereafter as common arithmetic.

[3] An element of S is called a prime if it cannot divide the product of two elements of S without dividing at least one factor.

[4] *Algebraischen Grössen*, Leipzig (1903), Chapters 1, 4.

[5] Math. Annalen 106 (1932), pp. 114–130; Math. Zeitschr. 37 (1933), pp. 39–60.

$a = b$ or if there exists an element c such that $ac = b$. We write then $a \mid b$. We observe that

(1) If $a_i \mid b_i$, $(i = 1, \cdots, k)$, then $a_1 \cdots a_k \mid b_1 \cdots b_k$.

(1) is untrue if multiplication is not commutative.

If $a \mid b$ and $b \mid a$, a and b are said to be associate, written $a \backsim b$. Non-associate elements are said to be distinct. If $a \mid x$ for every x in S, a is called a unit. Non-units are called integral elements. A divisor of an element is called proper if it is neither a unit nor an associate. Units have no proper divisors. An integral element with no proper divisors is called an irreducible. An element with only one distinct irreducible divisor is called a power of that divisor. The number of distinct proper divisors of a power increased by unity is called its multiplicity. We write $p^{(n)}$ for a power of the irreducible p of finite multiplicity n. Thus $p^{(1)} \backsim p$.

An element a of S is called indecomposable if in every decomposition of a into a product of two or more factors, one of the factors is associate to a. An irreducible is necessarily indecomposable. To avoid trivialities, we shall assume

P 5 *S contains at least one integral element.*

We shall call a system which satisfies the five postulates P 1 – P 5 a band.[6]

3°. The next four postulates complete our definition of S.

P 6 *Every element of S has only a finite number of distinct divisors.*

P 7 *Two powers of the same irreducible are either equivalent or else one divides the other.*

P 8 *If an element of S is divisible by two distinct irreducibles, it is decomposable.*

P 9 *If an element of S is divisible by a number of other elements each of which is a power of a different distinct irreducible, then it is divisible by their product.*

We shall call any system satisfying all nine of our postulates an abstract arithmetic. The simplest such systems are common arithmetic with our operation interpreted either as addition, multiplication or the operation of finding a least common multiple of two or more integers.

All of our postulates save P 5 and P 9 are used by Fritz Klein in defining his "B-Menge."[7] But P 9 is also true in a B-Menge since multiplication there is idempotent. *Therefore every B-Menge containing integral elements is an abstract arithmetic.*

J. Koenig[8] and A. H. Clifford define a power of p of multiplicity n as $p \cdot p \cdots p$ to n factors. But since factorization is unique in both systems, a power of p in their sense will be a power of p in our sense, and P 7 and P 9 will be both satisfied in each system. P 6 is a postulate in Koenig's system, and P 8 is satisfied because Koenig's system is a semi-group. In Clifford's system P 6 is replaced by the weaker "Teilerketten" condition, but it is true by virtue

[6] This convenient term was suggested by Dr. A. H. Clifford.

[7] Math. Zeitschr., place cited, especially pp. 47–54.

[8] J. Koenig's system is identical with F. Klein's "A-Menge." (Math. Zeitschr., volume cited, pp. 42–47.)

of the unique factorization. P 8 obviously holds since no indecomposables save units appear in Clifford's system. Hence: *A band satisfying the Clifford conditions for unique decomposition*[9] *is an abstract arithmetic. A commutative semigroup*[10] *in which* P 6 *holds and in which all irreducibles are primes is an abstract arithmetic.*

4°. It follows from P 5 and P 6 that S contains at least one irreducible, and that every integral element has at least one irreducible divisor. Furthermore, every power is of finite multiplicity, and from P 7 it readily follows that the m distinct divisors of $p^{(m)}$ are equivalent to $p^{(1)}, p^{(2)}, \cdots, p^{(m)}$. Furthermore, $p^{(n)} \mid p^{(m)}$ when and only when $n \leqq m$.

Consider now any integral element a of S. If a is a power, it is uniquely representable in the form $a \backsim p^{(n)}$. In the contrary case, we deduce from P 8, P 6 and P 7 that a decomposition of a exists of the form

$$a \backsim p_1^{(\alpha_1)} p_2^{(\alpha_2)} \cdots p_k^{(\alpha_k)} \tag{2}$$

where $p_1, p_2, \cdots, p_k$ are distinct irreducibles.

In discussing such decompositions, it is convenient to allow powers to have the superscript zero, with the understanding that such a power is to be omitted from the decomposition.[11] Thus, for example, $a \backsim p_1^{(2)} p_2^{(4)} p_3^{(0)} p_4^{(0)} \backsim p_1^{(0)} p_2^{(0)} p_3^{(1)} p_4^{(1)}$ is to be taken merely as another way of writing $a \backsim p_1^{(2)} p_2^{(4)} \backsim p_3^{(1)} p_4^{(1)}$.

With this understanding, let $p_1, p_2, \cdots, p_k$ now stand for all the distinct irreducibles of S which divide a. These are finite in number by P 6. Suppose that a has in all s distinct decompositions of the form (2),

$$a \backsim p_1^{(\alpha_1)} p_2^{(\alpha_2)} \cdots p_k^{(\alpha_k)} \backsim p_1^{(\beta_1)} p_2^{(\beta_2)} \cdots p_k^{(\beta_k)} \backsim \cdots \backsim p_1^{(\lambda_1)} p_2^{(\lambda_2)} \cdots p_k^{(\lambda_k)} \tag{3}$$

when the superscripts $\alpha, \beta, \cdots, \lambda$ are now positive integers or zero. The number s of such sets is finite by P 6, since $p^{(n)}$ and $p^{(m)}$ are distinct if $n \neq m$.

Let $\mu_i = \max(\alpha_i, \beta_i, \cdots, \lambda_i), \qquad (i = 1, \cdots, k)$.

Then $p_i^{(\mu_i)} \mid a$. Hence by P 9, $p_1^{(\mu_1)} \cdots p_k^{(\mu_k)} \mid a$. But $p_i^{(\alpha_i)} \mid p_i^{(\mu_i)}$, $(i = 1, \cdots, k)$. Therefore, by (1),

$$p_1^{(\alpha_1)} p_2^{(\alpha_2)} \cdots p_k^{(\alpha_k)} \mid p_1^{(\mu_1)} p_2^{(\mu_2)} \cdots p_k^{(\mu_k)}.$$

Hence

$$a \backsim p_1^{(\mu_1)} p_2^{(\mu_2)} \cdots p_k^{(\mu_k)}. \tag{4}$$

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[9] Clifford, p. 328, Theorem 1.

[10] This system is merely a band with the additional condition that $ab = ac$ when and only when $b = c$.

[11] If S contains units, we would define $p^{(0)}$ as an associate to a unit. The simplification resulting from adjoining a unit or an identity element to the abstract system is lost in the applications to ring theory where the ring need contain no units, and the additive properties of the adjoined elements must be considered.

Thus the decomposition (4) appears among the set (3). We shall call it the canonical decomposition of a.

5°. If $b \backsim q_1^{(\nu_1)} q_2^{(\nu_2)} \cdots , q_e^{(\nu_e)}$, it is clear that necessary and sufficient conditions that $b \mid a$ are that every q be a p and that the multiplicity of each $q^{(\nu)}$ be less than or equal to the multiplicity of the corresponding $p^{(\mu)}$. Hence the divisors of a are given by the set of elements $p_1^{(n_1)} p_2^{(n_2)} \cdots p_k^{(n_k)}$, $(n_i = 0, 1, \cdots , \mu_i; i = 1, \cdots , k)$.

Given two integral elements c and d, let $s_1, s_2, \cdots , s_m$ be the distinct irreducible elements dividing either or both of them. With the convention adopted in section 4°, we may write their canonical decompositions in the form

$$c \backsim s_1^{(\gamma_1)} \cdots s_m^{(\gamma_m)} , \qquad d \backsim s_1^{(\delta_1)} \cdots s_m^{(\delta_m)} .$$

Then if $\theta_i = \max (\gamma_i, \delta_i)$, $\zeta_i = \min (\gamma_i, \delta_i)$, $(i = 1, \cdots , k)$ it is clear that

$$[c, d] \backsim s_1^{(\theta_1)} \cdots s_m^{(\theta_m)} , \qquad (c, d) \backsim s_1^{(\zeta_1)} \cdots s_m^{(\zeta_m)}$$

are the least common multiple and greatest common divisor of c and d in the sense of common arithmetic. Moreover $c \backsim [s_1^{(\gamma_1)}, s_2^{(\gamma_2)}, \cdots , s_m^{(\gamma_m)}]$.

If we assume that S contains a unit, so that c and d will always have a common divisor, it is easily verified that the operations $[x, y]$ and (x, y) just defined satisfy Klein's postulates for a "Sternverband."[12] Hence even if we do not have unique decomposition with respect to multiplication, we always have unique decomposition with respect to the least common multiple operation.

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[12] Annalen paper already cited.

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A SUBSTITUTE FOR THE EUCLID ALGORITHM IN ALGEBRAIC FIELDS*

BY C. C. MACDUFFEE AND E. D. JENKINS

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1. **Introduction.** Most attempts to develop a greatest common divisor process for algebraic fields have been by a generalization of the algorithm of Euclid.[1] Even in the quadratic case only 15 fields are known having Euclid algorithms based on the principle of decreasing norms.[2] A few other fields have been treated by special devices. The vast majority of quadratic fields of class number unity, to say nothing of higher fields, have no known greatest common divisor process patterned after Euclid's algorithm.

A method for determining the greatest common ideal divisor of two ideals in an algebraic field by the use of integral matrices is known.[3] It is the purpose of the present paper to demonstrate the practicability of this method, and to show that it leads to a greatest common divisor of two or more numbers whenever this exists, even when the class number of the field is not unity. The computational limitations of the method are determined by the possibility of solving a certain diophantine equation. For a quadratic field this is a binary quadratic, and the theory is complete. For fields of higher degree the method is only tentative.

It seems doubtful that there are many problems now solved by means of a Euclid algorithm which cannot be as easily treated by means of any greatest common divisor process. Certainly in the theory of matrices this seems to be true.[4]

2. **Quadratic fields.** The integral numbers of $\mathfrak{F}(\sqrt{m})$ are given by $x + y\theta$ where $\theta^2 = m$ if $m \equiv 2, 3 \pmod 4$ (Case I), and $\theta^2 = \theta + m'$, $4m' = m - 1$ if $m \equiv 1 \pmod 4$ (Case II). We define for the respective cases[5]

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \Gamma = \begin{bmatrix} 0 & 1 \\ m & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ m' & 1 \end{bmatrix}.$$

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[1] R. Dedekind, *Werke*, Vol. 2, p. 373.
H. Hasse, *J. reine angew. Math.* 159 (1928), pp. 3–12.
W. Krull, *Math. Ann.* 105 (1931), pp. 1–14.
J. Wedderburn, *J. reine angew. Math.* 167 (1932), pp. 129–141.

[2] O. Perron, *Math. Ann.* 107 (1932), pp. 489–495.

[3] C. C. MacDuffee, *Bull. Amer. Math. Soc.* 37 (1931), pp. 841–853.

[4] *Bull. Amer. Math. Soc.* 39 (1933), pp. 564–584.

[5] *Annals of Math.* II, 29 (1928), p. 200.

The integral numbers $\xi = x + y\theta$ of $\mathfrak{F}$ are isomorphic with the matrices $S(\xi) = xI + y\Gamma$.

If

$$S(\xi) = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix},$$

then

$$\omega_1 = g_{11} + g_{12}\theta, \qquad \omega_2 = g_{21} + g_{22}\theta$$

constitute a minimal basis[6] for the principal ideal $\{\xi\}$.

If ξ_1 and ξ_2 are two integral numbers of $\mathfrak{F}$, the matrices $S(\xi_1)$, $S(\xi_2)$ have a greatest common right divisor

$$D = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}$$

which is easily calculated.[7] Then the numbers

$$\eta_1 = d_{11} + d_{12}\theta, \qquad \eta_2 = d_{21} + d_{22}\theta$$

are a minimal basis[8] for the greatest common ideal divisor of $\{\xi_1\}$ and $\{\xi_2\}$.

Now D is the left associate of some matrix $S(\delta)$ if and only if δ is a greatest common divisor of ξ_1 and ξ_2. If $\mathfrak{F}$ is of class number unity, this matrix $S(\delta)$ always exists unless $\xi_1 = \xi_2 = 0$. If $\mathfrak{F}$ contains more than one class, it may or may not exist. There are as many determinations of δ as there are units in $\mathfrak{F}$.

We wish to determine all unimodular matrices U such that UD is the S-matrix of some number δ of $\mathfrak{F}$. A matrix is an S-matrix if and only if it is commutative with Γ.[9] Thus we wish all unimodular matrices U such that

$$UD\Gamma = \Gamma UD, \qquad UD\Gamma D^{\mathrm{I}} = \Gamma U.$$

But $D\Gamma D^{\mathrm{I}} \equiv T$ has integral elements, since D corresponds to an ideal.[10] Then

$$UT = \Gamma U.$$

If U^{A} denotes the adjoint of U, then

$$U^{\mathrm{A}}\Gamma = TU^{\mathrm{A}}, \tag{1}$$

and U^{A} determines U.

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[6] *Trans. Amer. Math. Soc.* 31 (1929), p. 77.

[7] A. Châtelet, *Groupes abéliens finis*, Paris 1924. For an exposition, see *Ergeb. Math.* II—5 (1933), p. 35.

[8] *Math. Ann.* 105 (1931), p. 663.

[9] *Trans. Amer. Math. Soc.* 31 (1929), p. 78 and p. 87.

[10] *Trans. Amer. Math. Soc.* 31 (1929), p. 76.

All matrices B such that

$$B\Gamma = TB \tag{2}$$

form a minor class of ideal matrices, and hence are all of the form

$$vB_1 + wB_2$$

where B_1 and B_2 are fixed.[11] In fact, B_1 and B_2 may be chosen in the form

$$B_1 = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & q_{12} \\ q_{21} & q_{22} \end{bmatrix}$$

where p_{11} and q_{21} are positive and minimal. Hence B is determined by its first column. This means that the four equations for the elements b_{ij} of B arising from (2) are equivalent to but two equations of the form

$$\begin{aligned} b_{12} &= a_{11}b_{11} + a_{12}b_{21}, \\ b_{22} &= a_{21}b_{11} + a_{22}b_{21} \end{aligned} \tag{3}$$

where b_{11} and b_{21} are arbitrary. The a's are rational but not necessarily integral.

The determination of B_1 and B_2 is a tentative process, but it requires only a finite number of trials, and in practice proves to be quite simple. If d is the least common denominator of the a's, only positive integral values of b_{11}, $b_{21} \leqq d$ need be tried. A pair of values making b_{12}, b_{22} integers and b_{11} as small as possible but positive determines B_1. The smallest positive b_{21}, with $b_{11} = 0$, which makes b_{12}, b_{22} integers determines B_2.

The next step is to determine those matrices U^{A} which are of the form

$$vB_1 + wB_2$$

and are unimodular. This leads to the theory of quadratic forms, and the complete solution can be obtained.[12]

For every U^{A} so found, we find U and $UD = S(\delta)$. From this, a greatest common divisor δ of ξ_1 and ξ_2 is evident.

For example, let it be required to find the greatest common divisors of

$$\xi_1 = 68 + 5\sqrt{33}, \qquad \xi_2 = 31 + \sqrt{33}$$

in the field $\mathfrak{F}(\sqrt{33})$. A basis for this field is $1, \theta = \frac{1}{2}(1 + \sqrt{33})$, so

$$\xi_1 = 63 + 10\theta, \qquad \xi_2 = 30 + 2\theta.$$

$$\Gamma = \begin{bmatrix} 0 & 1 \\ 8 & 1 \end{bmatrix}, \qquad S(\xi_1) = \begin{bmatrix} 63 & 10 \\ 80 & 73 \end{bmatrix}, \qquad S(\xi_2) = \begin{bmatrix} 30 & 2 \\ 16 & 32 \end{bmatrix}.$$

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[11] *Trans. Amer. Math. Soc.* 31 (1929), p. 85. Since $\mathfrak{F}$ is a field, its discriminant $|c_{rsl}| \neq 0$ See *Trans. Amer. Math. Soc.* 33 (1931), p. 427.

[12] E.g., see L. E. Dickson, *Introduction to the theory of numbers*, Chicago 1929.

We find that

$$D = \begin{bmatrix} 1 & 2 \\ 0 & 29 \end{bmatrix}$$

is a g. c. r. d. of $S(\xi_1)$ and $S(\xi_2)$. Then

$$T = D\Gamma D^{\mathrm{I}} = \begin{bmatrix} 16 & -1 \\ 232 & -15 \end{bmatrix}$$

has integral elements, as predicted.

Equations (3) become

$$b_{12} = 2b_{11} - \tfrac{1}{8}b_{21},$$

$$b_{22} = 29b_{11} - \tfrac{15}{8}b_{21}.$$

From this we find our basis matrices to be

$$B_1 = \begin{bmatrix} 1 & 2 \\ 0 & 29 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & -1 \\ 8 & -15 \end{bmatrix}.$$

A solution of the diophantine equation

$$|vB_1 + wB_2| = 29v^2 - 31vw + 8w^2 = \pm 1. \tag{4}$$

is $v = -1, w = -2$.

Thus

$$U^{\mathrm{A}} = -B_1 - 2B_2 = \begin{bmatrix} -1 & 0 \\ -16 & 1 \end{bmatrix}, \qquad U = \begin{bmatrix} 1 & 0 \\ 16 & -1 \end{bmatrix},$$

$$UD = \begin{bmatrix} 1 & 2 \\ 16 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2\begin{bmatrix} 0 & 1 \\ 8 & 1 \end{bmatrix} = S(1 + 2\theta).$$

Hence $\delta = 1 + 2\theta = 2 + \sqrt{33}$ is a greatest common divisor of ξ_1 and ξ_2, and since the fundamental unit of the field is $23 + 4\sqrt{33}$, every g. c. d. is of the form

$$\pm(23 + 4\sqrt{33})^k(2 + \sqrt{33}), \qquad (k = 0, \pm 1, \pm 2, \cdots).$$

By using undetermined coefficients it is easy to write

$$1 + 2\theta = (-3 - 2\theta)\xi_1 + (9 + 5\theta)\xi_2.$$

3. **Cubic fields.** With obvious modifications the theory carries over to cubic and higher fields. The practical limitation is set by our ability to solve the diophantine equation corresponding to (4). For fields of order higher than the quadratic, no general methods for solution are known. If one is satisfied

with a single greatest common divisor, this can often be found by guessing a solution of the diophantine equation.

Let us consider the cubic field $\mathfrak{F}(\theta)$ where $\theta^3 + 6\theta + 8 = 0$. This field is of class number 3, and has a basis $\epsilon_1 = 1$, $\epsilon_2 = \theta$, $\epsilon_3 = \frac{1}{2}\theta^2$.[13] We shall attempt to find a greatest common divisor of

$$\xi_1 = -10\epsilon_1 - 2\epsilon_2 + 14\epsilon_3, \qquad \xi_2 = 12\epsilon_1 + 11\epsilon_2 - \epsilon_3.$$

We find

$$\Gamma_2 = S(\epsilon_2) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ -4 & -3 & 0 \end{bmatrix}, \qquad \Gamma_3 = S(\epsilon_3) = \begin{bmatrix} 0 & 0 & 1 \\ -4 & -3 & 0 \\ 0 & -2 & -3 \end{bmatrix},$$

$$S(\xi_1) = \begin{bmatrix} -10 & -2 & 14 \\ -56 & -52 & -4 \\ 8 & -22 & -52 \end{bmatrix}, \qquad S(\xi_2) = \begin{bmatrix} 12 & 11 & -1 \\ 4 & 15 & 22 \\ -44 & -31 & 15 \end{bmatrix}.$$

It is a simple matter to find a greatest common right divisor

$$D = \begin{bmatrix} -10 & 0 & 0 \\ 4 & 1 & 0 \\ -2 & 0 & -1 \end{bmatrix}$$

of $S(\xi_1)$ and $S(\xi_2)$. Then

$$T_2 = D\Gamma_2 D^{\mathrm{I}} = \begin{bmatrix} -4 & -10 & 0 \\ 2 & 4 & -2 \\ 0 & 1 & 0 \end{bmatrix} \qquad T_3 = D\Gamma_3 D^{\mathrm{I}} = \begin{bmatrix} -2 & 0 & 10 \\ 0 & -3 & -4 \\ 1 & 2 & -1 \end{bmatrix}.$$

Equations

$$B\Gamma_2 = T_2 B, \qquad B\Gamma_3 = T_3 B$$

lead to a system of 18 equations which have the general solution

$$b_{12} = \tfrac{1}{2}b_{11} - \tfrac{5}{2}b_{31}, \qquad b_{13} = b_{11} + \tfrac{5}{2}b_{21},$$

$$b_{22} = \tfrac{3}{4}b_{21} + b_{31}, \qquad b_{23} = -\tfrac{1}{2}b_{11} - b_{21} + \tfrac{1}{2}b_{31},$$

$$b_{32} = -\tfrac{1}{4}b_{11} - \tfrac{1}{2}b_{21} + \tfrac{1}{4}b_{31}, \qquad b_{33} = -\tfrac{1}{4}b_{21}.$$

For $b_{11} = 1$, $b_{21} = 0$, $b_{31} = 1$, all the b's are integers, so we get B_1. For $b_{11} = 0$, $b_{21} = 4$, $b_{31} = 4$, we get B_2. For $b_{11} = b_{21} = 0$, $b_{31} = 4$, we get B_3. Thus

$$B_1 = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & -10 & 10 \\ 4 & 7 & -2 \\ 4 & -1 & -1 \end{bmatrix}, \qquad B_3 = \begin{bmatrix} 0 & -10 & 0 \\ 0 & 4 & 2 \\ 4 & 1 & 0 \end{bmatrix}.$$

[13] L. W. Reid, *Tafel der Klassenanzahlen für kubische Zahlkörper*, Göttingen 1899, p. 68.

Unless the ideal (ξ_1, ξ_2) is principal, the numbers ξ_1 and ξ_2 will have no greatest common divisor. In that case the diophantine equation

$$| x_1B_1 + x_2B_2 + x_3B_3 | = \pm 1$$

will have no solution. In this case it happens that B_1 is unimodular, so $(x_1, x_2, x_3) = (1, 0, 0)$ is a solution, and

$$U = B_1^{\wedge} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & -2 & 1 \end{bmatrix}.$$

Then

$$UD = \begin{bmatrix} 2 & 0 & 1 \\ -4 & -1 & 0 \\ 0 & -2 & -1 \end{bmatrix} = S(2\epsilon_1 + \epsilon_3) .$$

Thus $2\epsilon_1 + \epsilon_3 = 2 + \frac{1}{2}\theta^2$ is a greatest common divisor of ξ_1 and ξ_2, and may be written

$$2\epsilon_1 + \epsilon_3 = (-33\epsilon_1 + 9\epsilon_2 + 5\epsilon_3)\xi_1 + (-10\epsilon_1 - 2\epsilon_2 + 14\epsilon_3)\xi_2 .$$

All other greatest common divisors can be found as soon as a fundamental system of units for this field is obtained.

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CLASSIFICATION OF LIE ALGEBRAS

BY M. S. KNEBELMAN

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The problem before us has its origin in the theory of finite continuous groups of transformations. The usual method of classification depends on the existence or non-existence of invariant subgroups of different types so that we know—thanks to the work of Killing, Cartan and Weyl—all about simple and semi-simple groups on the one hand and integrable or solvable groups on the other. If the group does not belong to either of these classes we know very little about the different types except in the case of groups of comparatively small order.

In the following pages I propose to give an outline of a method of classification which does not depend so much on the order of the group as it does on its *genus*. Unfortunately the method does not apply or only partially applies in a good many cases. However it may be possible to further refine it so as to exclude at least some of the exceptional cases.

Another remark; the first part of what follows applies to finite linear algebras in general and I have already applied it to some simple cases of linear associative algebras. Recently Wedderburn gave the 25 distinct types of Boolean linear associative algebras of 2 elements among which there are just two of genus zero. It is these two that our method gives at once even if the algebra were of n elements.

In the present paper I consider only Lie algebras of n elements and genus less than $n - 2$.

1. **Nullity and Genus of a Group.** Let $X_1, X_2 \cdots X_n$ be the symbols of an infinitesimal Lie group of order n. Then we know that the commutators are linear combinations with constant coefficients of the symbols; these constants—the structure constants of the group—are numbers of the underlying number field which we assume to be the field of complex numbers, although this restriction may not be necessary.

Using the customary notation and summation convention we have

$$[X_i, X_j] = c_{ij}^k X_k \qquad (c_{ij}^k = -c_{ji}^k). \tag{1.1}$$

If we change the basis of our group to $\bar{X}_i$ given by

$$\bar{X}_i = a_i^j X_j, \qquad |a_i^j| \neq 0$$

then

$$[\bar{X}_i, \bar{X}_j] = \bar{c}_{ij}^k \bar{X}_k$$

where

$$\tilde{c}^k_{ij}\, a^h_k = c^h_{lm} a^l_i a^m_j . \tag{1.2}$$

That is, the structure constants of a group are the components of a mixed antisymmetric tensor.

We now consider the following construction. Suppose we select a set of $p(2 \leqq p \leqq n)$ linear combinations of the symbols of our group and form the $p(p-1)/2$ commutators. Among them we shall in general find a number which are independent of the ones with which we started and of each other. Say there are p_1 such symbols. Again we form all possible commutators of the $p + p_1$ symbols and get an additional set of p_2 new symbols and repeat this process indefinitely. One of two things must happen: at the end of at most $n - p$ steps we shall either get all symbols of the group or a subset of them which will be the symbols of a proper subgroup. If the structure of the group be such that it is possible to find a set of p symbols whose successive commutators generate the group but impossible to find less than p such symbols we say the nullity of the algebra is p. In other words

DEFINITION. *The nullity of a group is the least number of independent symbols whose successive commutators generate the group.* Obviously this definition applies to a perfectly general linear algebra except that in this case the nullity may be as small as 1. For a Lie algebra the nullity may be one only in the trivial case of a one parameter group.

It may be pointed out that Sylvester's law of nullity for matrices holds for algebras if by "product" we understand the direct product; in fact this is the reason for the name. To prove it suppose G_n of nullity p is the direct product of G_{n_1} and G_{n_2} of nullities p_1 and p_2 respectively. Let $X_1 \cdots X_{n_1}$ and $Y_1 \cdots Y_{n_2}$ be the symbols of the two groups and let $X_1 \cdots X_{p_1}$ and $Y_1 \cdots Y_{p_2}$ be the symbols that generate G_{n_1} and G_{n_2}. Then obviously $X_1, \cdots, X_{p_1}, Y_1, \cdots, Y_{p_2}$ will generate G_n but it may so happen that $a^\alpha_\sigma X_\alpha + b^\lambda_\sigma Y_\lambda$, $\alpha = 1, \cdots, p_1$; $\lambda = 1, \cdots, p_2$; $\sigma = 1, \cdots, p' < p$ will also generate it. On the other hand if G_n could be generated by less than p_1 or p_2 symbols so could G_{n_1} or G_{n_2} which is impossible since $[X, Y] = 0$. Hence we have

[1.1] The nullity of the direct product of two groups is not greater than the sum of the nullities of the factors and not less than the nullity of either factor.

As in the case of matrices it is usually more convenient to deal with the rank of a matrix, so in the case of a group we shall deal with its genus, given by

DEFINITION. *The genus of a group is the difference between its order and its nullity.*

We now consider the following problem: Given a group G_n of genus r; what must be the form of its structure? Let $X_1, \cdots, X_n$ be the basis of G_n; if the genus is to be r it must be true that if we choose $n - r - 1$ symbols

$$\bar{X}_\alpha = a^i_\alpha X_i , \qquad \alpha = 1, \cdots, n - r - 1$$

the matrix $\| a^i_\alpha \|$ being of rank $n - r - 1$, and form the successive commutators

of $\bar{X}_\alpha$ we must get at most $n-1$ independent symbols no matter how the matrix $\| a_\alpha^i \|$ is chosen.

To express these conditions symbolically we use the following notation

$$
\begin{aligned}
&[X_j, X_k] \equiv X_{jk} = c_{jk}^i X_i \\
&[[X_j, X_k], X_l] \equiv X_{jk,l} = c_{jk}^h c_{hl}^i \equiv c_{jk,l}^i X_i \qquad (1.3) \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&X_{jk,lm,p\cdots} = c_{jk,lm,p\cdots}^i X_i .
\end{aligned}
$$

Obviously $c_{jk,lm\cdots}^i$ are the components of a tensor for a change of basis. We also let

$$
\bar{c}_{\alpha\beta,\gamma\delta\cdots}^i = c_{jk,lm\cdots}^i a_\alpha^j a_\beta^k a_\gamma^l a_\delta^m \cdots . \qquad (1.4)
$$

Our problem then is to find the form of the structure constants c_{jk}^i in order that the matrix

$$
\left\| \begin{array}{l}
a_1^1 \cdots a_{n-r-1}^1 \; \bar{c}_{12}^1 \; \bar{c}_{13}^1 \cdots \bar{c}_{\alpha\beta}^1 \cdots \bar{c}_{\alpha\beta,\gamma}^1 \cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
a_1^n \cdots a_{n-r-1}^n \; \bar{c}_{12}^n \; \bar{c}_{13}^n \cdots \bar{c}_{\alpha\beta}^n \cdots \bar{c}_{\alpha\beta,\gamma}^n \cdots
\end{array} \right\|
$$

shall be of rank $n-1$ at most. This means that if we form all n rowed determinants in the above matrix each of them must vanish for all values of $n(n-r-1)$ variables a_α^i. This will give us a number of tensor equations that the tensor c_{jk}^i must satisfy and the most general solution of these equations—if any solution exists—is the required structure. It will then have to be verified that the genus is really r and not some smaller number since if the genus is $r-h$, all of the above conditions are still satisfied.

Obviously this method is too laborious even in the cases where n and r are comparatively small. We therefore look at the problem from a different point of view. Let the range $1, 2, \cdots, n$ be divided into two complementary ranges consisting of $n-r-1$ and $r+1$ indices. Let $\alpha, \beta, \cdots$ run over the first range and $\sigma, \tau \cdots$ over the second. Then

$$
\begin{aligned}
&X_{\alpha\beta} = c_{\alpha\beta}^\epsilon X_\epsilon + c_{\alpha\beta}^\sigma X_\sigma \\
&X_{\alpha\beta,\gamma} = c_{\alpha\beta,\gamma}^\epsilon X_\epsilon + c_{\alpha\beta,\gamma}^\sigma X_\sigma \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&X_{\alpha\beta,\gamma_1\cdots\gamma_k} = c_{\alpha\beta,\gamma_1\cdots\gamma_k}^\epsilon X_\epsilon + c_{\alpha\beta,\gamma_1\cdots\gamma_k}^\sigma X_\sigma .
\end{aligned}
$$

If the genus is to be r, the symbols $c_{\alpha\beta,\gamma_1\cdots\gamma_k}^\sigma$ must depend on at most r of them, that is, the matrix of $r+1$ rows and $(n-r-1)^{k+2}$ columns whose elements are $c_{\alpha\beta,\gamma_1\cdots\gamma_k}^\sigma$ must be of rank r at most. Hence

$$
c_{\alpha\beta,\gamma_1\cdots\gamma_k}^\sigma = \sum_{\lambda=1}^{r-h} p_{\alpha\beta,\gamma_1\cdots\gamma_k}^{(\lambda)} s_{(\lambda)}^\sigma
$$

and hence, in particular,

$$c^{\sigma}_{\alpha\beta} = p^{(\lambda)}_{\alpha\beta} \cdot s^{\sigma}_{(\lambda)} \qquad \lambda = 1, \cdots, r-h .$$

This must be true no matter how the range $1, \cdots, n$ is divided; therefore

$$c^{i}_{jk} = p^{(\lambda)}_{jk} \cdot s^{i}_{(\lambda)} \qquad i \neq j, k . \tag{1.5}$$

Since c^{i}_{jk} is a tensor—λ is not a tensor index—we see that $p^{(\lambda)}_{jk}$ may be taken to be a set of $r-k$ skew-symmetric tensors and $s^{i}_{(\lambda)}$ a set of contravariant vectors. In order to obtain the remaining part of the structure, that is, the form of c^{i}_{jk} when $i = j$ or k we prove the following.

LEMMA. If T^{i}_{jk} is a tensor whose components for $i \neq j, k$ vanish in every coordinate system, then

$$T^{i}_{jk} = \frac{1}{n^2-1}\{\delta^{i}_{j}(n T^{h}_{hk} - T^{h}_{kh}) + \delta^{i}_{k}(n T^{h}_{jh} - T^{h}_{hj})\} . \tag{1.6}$$

Although it is only necessary for (1.6) that $T^{1}_{22} = T^{1}_{23} = 0$ in every coordinate system, we assume $T^{i}_{jk} = 0$ if $i \neq j, k$ and apply an arbitrary non-singular transformation to our tensor; we then have

$$T^{\alpha}_{\beta\gamma} u^{\beta}_{j} u^{\gamma}_{k} \bar{u}^{i}_{\alpha} \equiv 0 \qquad \text{for all values of } u^{\beta}_{j} .$$

Differentiating this set of equations with respect to u^{ϵ}_{h}, we obtain

$$T^{\alpha}_{\beta\gamma}[u^{\beta}_{j} u^{\gamma}_{k}(-\bar{u}^{i}_{\epsilon}\bar{u}^{h}_{\alpha}) + \delta^{\beta}_{\epsilon}\delta^{h}_{j} u^{\gamma}_{k}\bar{u}^{i}_{\alpha} + u^{\beta}_{j}\delta^{\gamma}_{\epsilon}\delta^{h}_{k}\bar{u}^{i}_{\alpha}] \equiv 0$$

and letting $u^{i}_{\alpha} = \delta^{i}_{\alpha}$ we get

$$T^{h}_{jk}\delta^{i}_{l} = T^{i}_{lk}\delta^{h}_{j} + T^{i}_{jl}\delta^{h}_{k} ; \qquad h, l \text{ arbitrary } i \neq j, k . \tag{1.7}$$

From this it follows that

$$T^{h}_{jk} = T^{i}_{ik}\delta^{h}_{j} + T^{i}_{ji}\delta^{h}_{k} \qquad i \text{ not summed.} \tag{1.8}$$

Hence $\quad T^{j}_{jk} = T^{i}_{ik}; \quad T^{k}_{kk} = T^{i}_{ik} + T^{i}_{ki} \qquad i, j, k \text{ not summed and} \neq .$

From the above we have

$$T^{j}_{jk} = T^{1}_{1k} = T^{2}_{2k} = \cdots T^{k-1}_{k-1k} = T^{k+1}_{k+1k} = \cdots T^{n}_{nk}$$
$$= \frac{T^{h}_{hk} - T^{k}_{kk}}{n-1} \qquad (k \text{ not summed})$$

or

$$n T^{j}_{jk} + T^{j}_{kj} = T^{h}_{hk} .$$

Similarly

$$T^{j}_{jk} + n T^{j}_{kj} = T^{h}_{kh} .$$

Hence

$$T^{j}_{jk} = \frac{n T^{h}_{hk} - T^{h}_{kh}}{n^2-1} ; \qquad T^{j}_{kj} = \frac{n T^{h}_{kh} - T^{h}_{hk}}{n^2-1} ,$$

and when these are substituted into (1.8) we get (1.6). If the tensor is skew symmetric (1.6) reduce to

$$T^i_{jk} = \frac{1}{n-1}\left(\delta^i_j T^h_{hk} - \delta^i_k T^h_{hj}\right);$$

T^h_{hj} being a covariant vector, we denote it by $(n-1)u_j$. Hence T^i_{jk} in this case is of the form

$$T^i_{jk} = \delta^i_j u_k - \delta^i_k u_j . \tag{1.9}$$

Returning to our group, we found in (1.5) that $c^i_{jk} - p^{(\lambda)}_{jk} \cdot s^i_{(\lambda)}$ is a tensor whose components for $i \neq j, k$ vanish in every coordinate system; hence according to our lemma we must have

$$c^i_{jk} = a^{(\lambda)}_{jk} s^i_{(\lambda)} + \delta^i_j u_k - \delta^i_k u_j , \qquad \lambda = 1, \cdots, r-h . \tag{1.10}$$

So far we have used only the fact that $n - r - 1$ elements will not generate the group. Suppose $h > 0$ and we choose any $n - r - 1 + h$ elements $a^i_\alpha X_i = \bar{X}_\alpha$, $\alpha = 1; \cdots, n - r - 1 + h$. Then

$$\begin{aligned}[\bar{X}_\alpha, \bar{X}_\beta] &= a^j_\alpha a^k_\beta \left(p^\lambda_{jk} s^i_\lambda X_i + X_j u_k - X_k u_j\right) \\ &= \bar{p}^\lambda_{\alpha\beta} s^i_\lambda X_i + \bar{X}_\alpha \bar{u}_\beta - \bar{X}_\beta \bar{u}_\alpha \\ &\equiv 0 \pmod{\bar{X}_\alpha;\ s^i_\lambda X_i} .\end{aligned}$$

Similarly $[s^i_\lambda X_i, \bar{X}_\alpha] = [s^i_\lambda X_i, s^j_\mu X_j] \equiv 0 \pmod{\bar{X}_\alpha;\ s^i_\lambda X_i}$. Hence no $n - r - 1 + h$ elements will generate the group and the genus is less than r. We can therefore conclude that if a group G_n is of genus r its structure must be

$$c^i_{jk} = p^\lambda_{jk} S^i_\lambda + \delta^i_j U_k - \delta^i_k u_j \qquad \lambda = 1, \cdots, r , \tag{1.11}$$

where the matrices $\| p^\lambda_{jk} \|$ and $\| s^i_\lambda \|$ are each of rank r. The converse of the above statement need not be true; if a group has the structure given by (1.11) obviously the genus can not be greater than r but if it so happens that

$$p^\lambda_{jk} s^i_\lambda = q^\alpha_{jk} t^i_\alpha + \delta^i_j v_k - \delta^i_k v_j \qquad \begin{matrix} \lambda = 1 \cdots r \\ \alpha = 1, \cdots, r' < r \end{matrix}$$

then the genus may be as small as r'. If then the structure constants of a group are given and we want to determine its genus, we must find the least value of r for which (1.11) may be solved for the unknowns p, s and u or we may turn back to the matrix whose elements are given by (1.4) and see which commitants of the group vanish.

We may state the above results as the following theorem:

[1.2] If a Lie group of order n is of genus $r < n - 2$, its structure constants have the form

$$c^i_{jk} = p^\lambda_{jk} s^i_\lambda + \delta^{ih}_{jk} u_h \qquad \lambda = 1, \cdots, r ,$$

where s_λ^i is a set of r independent contravariant vectors, u_j a covariant vector—or zero—and p_{jk}^λ a set of r independent, skew-symmetric covariant tensors. If the genus is zero the structure constants are

$$c_{jk}^i = \delta_{jk}^{ih} u_h . \tag{1.12}$$

Conversely if the structure of a group is given by (1.11) and if it can not be written in that form with $\lambda = 1, \cdots, r' < r$ then the genus of the group is r.

2. **Jacobi's Conditions.** The constants of structure of a Lie group besides being skew-symmetric must also satisfy Jacobi's conditions

$$c_{jk}^h c_{hl}^i + c_{kl}^h c_{hj}^i + c_{lj}^h c_{hk}^i = 0 . \tag{2.1}$$

If the order of the group is one or two, these conditions are satisfied identically and since a two parameter group can not be generated by one symbol we have [2.1] Every one and two parameter group is of genus zero.

We shall therefore confine our attention to groups of order $n > 2$ and genus $r < n - 2$. When the values of the structure constants obtained in §1 are substituted into (2.1) we get

$$\begin{aligned} s_\mu^l \{ s_\lambda^h [p_{ij}^\lambda p_{hk}^\mu + p_{jk}^\lambda p_{hi}^\mu + p_{ki}^\lambda p_{hj}^\mu] - (p_{ij}^\mu u_k + p_{jk}^\mu u_i + p_{ki}^\mu u_j) \} \\ = s_\lambda^h u_h (p_{ij}^\lambda \delta_k^l + p_{jk}^\lambda \delta_i^l + p_{ki}^\lambda \delta_j^l) . \end{aligned} \tag{2.2}$$

If $n = 3$, the only case we can consider is $r = 0$ which we leave aside for the present. Let $n > 3$ then we can choose $l \neq i, j, k$ and we then have

$$s_\mu^l \{ P_{ijk}^\mu \} = 0 , \tag{2.3}$$

where P_{ijk}^μ has the significance of the brace in (2.2). Since $\| s_\mu^l \|$ is of rank r, a basis of the group can be found such that no r rowed determinant of the above matrix vanishes; therefore $P_{ijk}^\mu = 0$ for every choice of i, j, k. Consequently Jacobi's conditions reduce to

$$\begin{cases} s_\lambda^h [p_{ij}^\lambda p_{hk}^\mu + p_{jk}^\lambda p_{hi}^\mu + p_{ki}^\lambda p_{hj}^\mu] = p_{ij}^\mu u_k + p_{jk}^\mu u_i + p_{ki}^\mu u_j \\ (p_{ij}^\lambda \delta_k^l + p_{jk}^\lambda \delta_i^l + p_{ki}^\lambda \delta_j^l) s_\lambda^h u_h = 0 . \end{cases} \tag{2.4}$$

If in the last set of (2.4) we choose $l = k \neq i, j$ we get $p_{ij}^\lambda (s_\lambda^h u_h) = 0$ and since $\| p_{ij}^\lambda \|$ is of rank r we have

$$s_\lambda^h u_h = 0 . \tag{2.5}$$

Therefore Jacobi's conditions reduce to the first set in (2.4) and (2.5). Whether solutions of these equations exist and what is their canonical form will be considered in the next section. For the present we see since $\| s_\lambda^h \|$ is of rank r, a basis exists for which

$$s_\lambda^h = \delta_\lambda^h \qquad u_h = \delta_h^n \quad \text{if} \quad u_h \neq 0 .$$

Therefore the structure may be reduced to

$$c^i_{jk} = p^{(\lambda)}_{jk}\delta^i_{(\lambda)} + \delta^i_j\delta^{(n)}_k - \delta^i_k\delta^{(n)}_j \tag{2.6}$$

or

$$c^i_{jk} = p^{(\lambda)}_{jk}\delta^i_{(\lambda)} \qquad \text{if} \quad u_h = 0 .$$

Hence

$$[X_j, X_k] = h^\lambda_{jk}X_\lambda + X_j\delta^n_k - X_k\delta^n_j$$

$$[X_\mu, X_\nu] = p^\lambda_{\mu\nu}X_\lambda; \qquad [X_\mu, X_j] = p^\lambda_{\mu j}X_\lambda + X_\mu\delta^n_j .$$

Therefore $X_1, \cdots, X_r$ are the symbols of an r parameter invariant subgroup. As we prove in the next section, every group of genus zero has an invariant abelian subgroup of order $n - 1$ and since a simple group has no proper invariant subgroup, we have

[2.2] Every simple group of order n is of genus $n - 2$, that is, in every simple group a pair of symbols may be found whose successive commutators generate the whole group.

From (2.6) we see that the derived group is of order $n - 1$ at most; hence a group with the structure (2.6) can not be semi-simple. Therefore [2.2] holds for semi-simple groups; in fact it holds for all "perfect" groups, a perfect group being defined as one whose derived group is the group itself.

3. **Groups of genus zero.** In this case there are only two possibilities

(i) $u_k = 0$; then $[X_j, X_k] = 0$ and the group is Abelian.

(ii) $u_k \neq 0$; we can find a basis for which $u_k = \delta^n_k$; hence

$$[X_j, X_k] = 0 \qquad [X_j, X_n] = X_j \qquad j, k < n . \tag{3.1}$$

In this group there is a series of composition

$$G_n , G_{n-1} , 1;$$

therefore the group is solvable. This group may be realized by the finite transformations

$$\bar{x}^i = e^{-a^n}(x^i + a^i x^n)$$

which carry the hyperplanes $x^n = \text{const.}$ into such hyperplanes, the points in the hyperplane being transformed by a dilation.

The conditions that the structure constants of a group of genus zero must satisfy may be obtained from (1.12) by eliminating u_j; the resulting equations are

$$c^i_{jk} - \frac{1}{n-1}\delta^i_j c^h_{hk} + \frac{1}{n-1}\delta^i_k c^h_{hj} = 0 . \tag{3.2}$$

The left hand side of (3.2) in a group of any genus defines a commitant of the structure, which we denote by Π^i_{jk}. There is a very close analogy between

these quantities and the projective connection of an affinely connected space. For this reason we may call a group of genus zero "projectively abelian" and say that two groups are "projectively isomorphic"—by isomorphism we mean simple isomorphism—if they have the same "projective structure." In other words; we say two groups are projectively isomorphic if there exists a vector v_i such that

$$\bar{c}^i_{jk} = c^i_{jk} + \delta^i_j v_k - \delta^i_k v_j \,. \tag{3.3}$$

If v_k is eliminated from (3.3) we get

$$\bar{c}^i_{jk} - \frac{1}{n-1}(\delta^i_j \bar{c}^h_{hk} - \delta^i_k \bar{c}^h_{hj}) = c^i_{jk} - \frac{1}{n-1}(\delta^i_j c^h_{hk} - \delta^i_k c^h_{hj})$$

that is $\bar{\Pi}^i_{jk} = \Pi^i_{jk}$. It is to be noted that Π^i_{jk} are in general not structure constants of a group, so that unlike the case of affine connections, a group being given there may not exist a projectively isomorphic group, the necessary and sufficient conditions for the existence of a projectively isomorphic group being the existence of a non-vanishing solution for the equations

$$\Pi^i_{jk} v_l + \Pi^i_{kl} v_j + \Pi^i_{lj} v_k = 0 \,. \tag{3.4}$$

If a given group G_n is of genus $r(< n-2)$, then the structure of any projectively isomorphic group is given by

$$\bar{c}^i_{jk} = p^\lambda_{jk} s^i_\lambda + \delta^i_j (u_k + v_k) - \delta^i_k (u_j + v_j) \,.$$

We therefore see that

[3.1] *If two groups are projectively isomorphic they are of the same genus (if* $r < n-2$).

From (3.4) by contraction we find $\Pi^h_{jk} v_h = 0$. Hence

[3.2] *If the rank of the matrix of the projective structure of a group* G_n *is* n, *projective isomorphism implies simple isomorphism.*

4. **Groups of genus one.** The structure of a group of genus one and order >3 is

$$c^i_{jk} = p_{jk} s^i + \delta^i_j u_k - \delta^i_k u_j \,. \tag{4.1}$$

We consider the two cases

1. $u_k = 0$. The Jacobi conditions in this case are

$$(p_{jk} p_{hl} + p_{kl} p_{hj} + p_{lj} p_{hk}) s^h = 0$$

and we may choose the basis of the group so that $s^h = \delta^h_1$. We then have

$$p_{jk} p_{1l} + p_{kl} p_{1j} + p_{lj} p_{1k} = 0 \,. \tag{4.2}$$

If $p_{ul} = 0$ these are satisfied identically and the Lie algebra is then given by

$$\begin{aligned} [X_j, X_k] &= p_{jk} X_1 \\ [X_1, X_k] &= 0 \end{aligned} \qquad j, k > 1 \,. \tag{4.3}$$

If $p_{11} \neq 0$ we may take $p_{12} = 1$; then if (4.2) hold for $j, k, 2$ they hold for all values of j, k, l. Hence we must have

$$p_{jk} = p_{1j}p_{2k} - p_{1k}p_{2j}$$

and we have

$$[X_1, X_2] = X_1; \qquad [X_1, X_k] = p_{1k}X_1; \qquad [X_2, X_k] = p_{2k}X_1; \tag{4.4}$$
$$[X_j, X_k] = (p_{1j}p_{2k} - p_{1k}p_{2j})X_1 \qquad j, h > 2.$$

2. $u_k \neq 0$. When the Jacobi conditions for this case are contracted we get

$$(n-1)\cdot u_h s^h \cdot p_{jk} = u_j p_{hk} s^h - u_k p_{hj} s^h = 0$$

since by (2.5) $u_h s^h = 0$. Hence

$$u_j p_{hk} s^h = u_k p_{hj} s^h \qquad \text{or} \qquad p_{hj} s^h = N u_j$$

and the Jacobi conditions reduce to

$$(N-1)(p_{jk}u_l + p_{kl}u_j + p_{lj}u_k) = 0 \tag{4.5}$$
$$u_h s^h = 0 \qquad p_{hj} s^h = N u_j.$$

If $N = 1$ we may choose a basis such that $s^h = \delta_1^h, u_h = \delta_h^n$; then $p_{1j} = 0, p_{1n} = 1$, p_{jk} = arbitrary numbers. Hence the multiplication table of the group is

$$[X_1, X_n] = 2X_1; \qquad [X_1, X_j] = 0; \qquad [X_j, X_k] = p_{jk}X_1; \tag{4.6}$$
$$[X_j, X_n] = p_{jn}X_1 + X_j, \qquad 1 < j, k < n.$$

If $N \neq 1$ we have as before $s^h = \delta_1^h$; $u_h = \delta_h^k p_{1k} = N\delta_k^n$ and

$$p_{jk}\delta_l^n + p_{kl}\delta_j^n + p_{lj}\delta_k^n = 0. \tag{4.7}$$

If $l = n$ $j, k \neq n$ we have $p_{jk} = 0$. Hence

$$[X_1, X_n] = (N+1)X_1, \qquad [X_1, X_j] = 0, \qquad [X_j, X_k] = 0, \tag{4.8}$$
$$[X_j, X_n] = p_{jn}X_1 + X_j, \qquad 1 < j, k < n \qquad N \neq 1.$$

5. **Groups of genus >1.** As was shown in §2 a basis of G_n may be found, such that the structure has the form

$$c_{jk}^i = p_{jk}^\lambda \delta_\lambda^i + \delta_j^i \delta_k^n - \delta_k^i \delta_j^n \tag{5.1}$$

and then $X_1, \cdots, X_r$ are the symbols of an r-parameter invariant subgroup whose structure constants are $p_{\mu\nu}^\lambda$.

We shall show that if this subgroup is given, the rest of the structure may be obtained from Jacobi's conditions (2.4). Moreover a non-singular transformation of the subgroup G_r may be used to put it in canonical form without affecting

the rest of the structure of G_n. For let a_μ^λ be a non-singular transformation with inverse $\bar{a}_\mu^\lambda$. Let

$$\bar{X}_\lambda = a_\lambda^\mu X_\mu \qquad \bar{X}_\sigma = X_\sigma \qquad \begin{matrix} \lambda, \mu = 1, \cdots, r \\ \sigma, \tau = r+1, \cdots, n. \end{matrix} \tag{5.2}$$

Then

$$\bar{u}_k = a_k^i \delta_i^{(n)} = a_k^n = \delta_k^{(n)}$$

$$\bar{s}_{(\lambda)}^i = \bar{a}_k^i \delta_{(\lambda)}^k = \bar{a}_\lambda^i \qquad \text{or} \qquad \bar{s}_{(\lambda)}^\mu = \bar{a}_\lambda^\mu, \bar{s}_\lambda^\sigma = 0.$$

Hence if $\bar{c}_{\beta\gamma}^\alpha$ are the structure constants of G_r in canonical form we have

$$\bar{p}_{\beta\gamma}^\alpha = \bar{c}_{\beta\gamma}^\epsilon a_\epsilon^\alpha \tag{5.3}$$

and (2.4) being invariant become

$$\bar{p}_{\nu(k}^\mu p_{ij)}^\lambda \bar{s}_\lambda^\nu = \bar{p}_{(ij}^\mu \delta_{k)}^n.$$

If these be multiplied by $\bar{s}_\mu^\rho$ and we set $\bar{p}_{ij}^\lambda \bar{s}_\lambda^\nu = q_{ij}^\nu$ we get

$$q_{\lambda(k}^\mu q_{ij)}^\lambda = q_{(ij}^\mu \delta_{k)}^n \tag{5.4}$$

so that the structure (5.1) becomes

$$\begin{aligned} c_{jk}^{\prime\, i} &= \bar{p}_{jk}^\lambda \bar{s}_\lambda^i + \delta_j^i \delta_k^n - \delta_k^i \delta_j^n \\ &= q_{jk}^\nu a_\nu^\lambda \bar{a}_\lambda^i + \delta_j^i \delta_k^n - \delta_k^i \delta_j^n \end{aligned}$$

or

$$c_{jk}^i = q_{jk}^\lambda \delta_\lambda^i + \delta_j^i \delta_k^n - \delta_k^i \delta_j^n \tag{5.5}$$

subject to the conditions (5.4), $c_{\beta\gamma}^\alpha$ being the structure of the subgroup G_r in canonical form. If in (5.4) we let $i, j = \alpha, \beta$ they become

$$q_{\alpha\beta}^\lambda q_{\lambda k}^\mu + q_{\beta k}^\lambda q_{\lambda\alpha}^\mu + q_{k\alpha}^\lambda q_{\lambda\beta}^\mu = q_{\alpha\beta}^\mu \delta_k^n, \tag{5.6}$$

which are a set of linear equations for the determination of $q_{\lambda k}^\mu$. Again if we let $k = \alpha$ in (5.4) we get

$$q_{ij}^\lambda q_{\lambda\alpha}^\mu + q_{j\alpha}^\lambda q_{\lambda i}^\mu + q_{\alpha i}^\lambda q_{\lambda j}^\mu = q_{j\alpha}^\mu \delta_i^n + q_{\alpha i}^\mu \delta_j^n, \tag{5.7}$$

which are also linear in the unknowns q_{ij}^λ. Finally (5.4) with $i, j, k > r$ are additional conditions that must be satisfied by the solutions of (5.7). Thus the problem of constructing a G_n of genus r with a prescribed invariant subgroup G_r reduces to solutions of linear equations, except in the case where G_r is abelian for then (5.6) vanish identically and (5.7)—without the first term—must be used for the determination of $q_{\lambda i}^\mu$. It was by means of the above equations that all n-parameter groups of genus 2 were computed. In conclusion we shall give the structures of some groups of genus two which are solvable—as

all groups of genus 0, 1, 2 are solvable—and a group G_n of genus three which is not solvable.

(a) $G_{n,2}$

$$[X_j, X_k] = (a_{1j}a_{2k} - a_{1k}a_{2j})X_1 + (a_{1k}\delta_j^n - a_{1j}\delta_k^n)X_2 + X_j\delta_k^n - X_k\delta_j^n, \qquad a_{12} = 1;$$

this is a solvable group having the invariant subgroup $[X_1, X_2] = X_1$.

(b) $G_{n,2}$ with the following structure:

$$[X_1, X_2] = 0; \qquad [X_1, X_j] = p_{ij}(X_1 + c_1^2 X_2) + X_1\delta_j^n,$$

$$[X_2, X_j] = c_2^1(p_{1j} - \delta_j^n)X_1 - (p_{1j} - 2\delta_j^n)X_2,$$

$$[X_i, X_j] = p_{ij}(X_1 + c_1^2 X_2) + X_i\delta_j^n - X_j\delta_i^n \qquad i, j > 2,$$

$$c_2^1 c_1^2 = -1 \qquad p_{1j} \neq 0 \qquad \text{for some } j < n.$$

(c) $G_{n,3}$ $\quad [X_1, X_2] = X_3 \quad [X_2, X_3] = X_1 \quad [X_3, X_1] = X_2$

$$[X_i, X_j] = \sum_{\lambda=1}^{3} (p_{\mu i}^{\lambda} p_{\nu j}^{\lambda} - p_{\nu i}^{\lambda} p_{\mu j}^{\lambda} + p_{\mu i}^{\nu}\delta_j^n - p_{\nu j}^{\mu}\delta_i^n)X_\lambda + X_i\delta_j^n - X_j\delta_i^n$$

$$\mu \equiv \lambda + 1, \quad \nu \equiv \lambda + 2 \pmod 3$$

$p_{\beta i}^{\alpha}$ arbitrary except for $p_{\beta i}^{\alpha} + p_{\alpha i}^{\beta} = 0$.

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STÄCKEL SYSTEMS IN CONFORMAL EUCLIDEAN SPACE

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In a previous paper[1] the general conditions for separable systems of Stäckel for the Schrödinger wave equation were derived, and they were applied to the case of euclidean 3-space, yielding a simple geometrical characterization of these systems. The latter result, as was learned after publication, had been obtained by analytical processes by Weinacht[2] and by beautiful geometrical considerations by Blaschke.[3] The present paper solves the problem of Stäckel systems for conformal euclidean 3-space, and determines in particular these systems for spaces for which in cartesian coordinates the fundamental form is

$$\frac{dx^2 + dy^2 + dz^2}{f(r^2)}, \qquad r^2 = x^2 + y^2 + z^2.$$

Of interest in this connection is the Schrödinger equation of the helium atom proposed by Gronwall (§7) for which $f(r^2) = r$. It is shown that the coordinates are not separable in this case because of the form of V.

In §8 are given the possible forms of the functions H_i^2 appearing in the Schrödinger equation in three variables, which are solvable by separation of variables, when there is no additional requirement concerning the character of the space.

1. We have shown that a necessary and sufficient condition that the variables be separable in a Schrödinger equation

$$\frac{1}{H}\sum_i \frac{\partial}{\partial x_i}\left(\frac{H}{H_i^2}\frac{\partial \varphi}{\partial x_i}\right) + k^2(E - V)\varphi = 0, \qquad H = H_1 \cdot \cdots \cdot H_n, \tag{1.1}$$

is that the H's satisfy the conditions

$$\frac{\partial^2 \log H_i^2}{\partial x_j\, \partial x_k} - \frac{\partial \log H_i^2}{\partial x_j}\frac{\partial \log H_i^2}{\partial x_k} + \frac{\partial \log H_i^2}{\partial x_j}\frac{\partial \log H_j^2}{\partial x_k} + \frac{\partial \log H_i^2}{\partial x_k}\frac{\partial \log H_k^2}{\partial x_j} = 0 \qquad (j \neq k), \tag{1.2}$$

[1] *Separable systems of Stäckel*, Annals of Math. (2), vol. 35 (1934), pp. 284–305.

[2] *Über die bedingt-periodische Bewegung eines Massenpunktes*, Math. Annalen, vol. 91 (1924), pp. 279–299.

[3] *Eine Verallgemeinerung der Theorie der konfocalen F_2*, Math. Zeitsch., vol. 27 (1928), pp. 655–668.

and that

$$R_{ij} = 0 \qquad (i \neq j), \tag{1.3}$$

where R_{ij} are the components of the Ricci tensor.[4]

We inquire under what conditions the quantities

$$\bar{H}_i^2 = e^{2\sigma} H_i^2 \tag{1.4}$$

are the coefficients of the fundamental form of a euclidean n-space $\bar{V}_n$, in which case V_n with the fundamental form

$$H_1^2 dx_1^2 + \cdots + H_n^2 dx_n^2 \tag{1.5}$$

is conformally flat. For V_n we have[5]

$$R_{jiik} = \frac{H_i^2}{4}\left[2\frac{\partial^2 \log H_i^2}{\partial x_j\, \partial x_k} + \frac{\partial \log H_i^2}{\partial x_j}\frac{\partial \log H_i^2}{\partial x_k} - \frac{\partial \log H_i^2}{\partial x_j}\frac{\partial \log H_j^2}{\partial x_k} - \frac{\partial \log H_i^2}{\partial x_k}\frac{\partial \log H_k^2}{\partial x_j}\right]. \tag{1.6}$$

For $\bar{V}_n$ the expression on the right must be zero in the quantities $\bar{H}_i^2$ given by (1.4). This gives the conditions

$$2\frac{\partial^2 \log H_i^2}{\partial x_j\, \partial x_k} + \frac{\partial \log H_i^2}{\partial x_j}\frac{\partial \log H_i^2}{\partial x_k} - \frac{\partial \log H_i^2}{\partial x_j}\frac{\partial \log H_j^2}{\partial x_k} - \frac{\partial \log H_i^2}{\partial x_k}\frac{\partial \log H_k^2}{\partial x_j} + 4\left(\frac{\partial^2 \sigma}{\partial x_j\, \partial x_k} - \frac{\partial \sigma}{\partial x_j}\frac{\partial \sigma}{\partial x_k} - \frac{1}{2}\frac{\partial \sigma}{\partial x_j}\frac{\partial \log H_j^2}{\partial x_k} - \frac{1}{2}\frac{\partial \sigma}{\partial x_k}\frac{\partial \log H_k^2}{\partial x_j}\right) = 0. \tag{1.7}$$

For a general coordinate system in which the coefficients of the fundamental quadratic forms are in the relations

$$\bar{g}_{ij} = e^{2\sigma} g_{ij},$$

we have

$$\bar{R}_{ij} = R_{ij} + (n-2)(\sigma_{,ij} - \sigma_{,i}\sigma_{,j}) + g_{ij}[\Delta_2 \sigma + (n-2)\Delta_1 \sigma],$$

where $\sigma_{,ij}$ is the second covariant derivative of σ with respect to the g's and $\sigma_{,i} = \dfrac{\partial \sigma}{\partial x_i}$.[6]

For the present case this reduces for $i \neq j$ to $\sigma_{,ij} - \sigma_{,i}\sigma_{,j} = 0$, that is

$$\frac{\partial^2 \sigma}{\partial x_i\, \partial x_j} - \frac{\partial \sigma}{\partial x_i}\frac{\partial \sigma}{\partial x_j} - \frac{1}{2}\frac{\partial \sigma}{\partial x_i}\frac{\partial \log H_i^2}{\partial x_j} - \frac{1}{2}\frac{\partial \sigma}{\partial x_j}\frac{\partial \log H_j^2}{\partial x_i} = 0 \qquad (i \neq j) \tag{1.8}$$

[4] *Separable systems of Stäckel*, Annals of Mathematics, vol. 35 (1934), pp. 289, 291. Hereafter a reference to this paper will be of the form S. pp. 289, 291.

[5] R. G., p. 119; a reference of this sort is to the author's Riemannian Geometry.

[6] R. G., p. 90, equation (28.6).

From (1.2), (1.7) and (1.8) it follows that the functions H_i must satisfy the conditions

$$\frac{\partial^2 \log H_i^2}{\partial x_j \, \partial x_k} = 0 \qquad (i, j, k, \neq),$$

$$\frac{\partial \log H_i^2}{\partial x_j}\frac{\partial \log H_i^2}{\partial x_k} - \frac{\partial \log H_i^2}{\partial x_j}\frac{\partial \log H_j^2}{\partial x_k} - \frac{\partial \log H_i^2}{\partial x_k}\frac{\partial \log H_k^2}{\partial x_j} = 0. \tag{1.9}$$

We have seen[7] that there are for the case $n = 3$ the following normal forms of solutions of (1.2) and (1.9);

$$H_1 = \frac{1}{\varphi(x_1)}, \qquad H_2 = x_1, \qquad H_3 = \psi(x_1), \tag{1.10}$$

where φ and ψ are functions of x_1 at most;

$$H_1^2 = 1, \qquad H_2^2 = X_2\sigma_1(\sigma_2 + \sigma_3), \qquad H_3^2 = X_3\sigma_1(\sigma_2 + \sigma_3), \tag{1.11}$$

where σ_1 is a function of x_1 at most, X_2 and σ_2 of x_2 at most and X_3 and σ_3 of x_3 at most;

$$H_1^2 = H_3^2 = \varphi_1 + e\varphi_3, \qquad H_2^2 = \varphi_1\varphi_3, \tag{1.12}$$

where e is $+1$ or -1, and φ_1 and φ_3 are positive functions of x_1 and x_3 respectively;

$$H_i^2 = X_i(x_i - x_j)(x_i - x_k) \qquad (i, j, k = 1, 2, 3;\ i, j, k \neq), \tag{1.13}$$

where X_i is a function of x_i at most.

In order that a V_3 be conformally flat it is necessary and sufficient that

$$R_{ij,k} - R_{ik,j} + \tfrac{1}{4}(g_{ik}R_{,j} - g_{ij}R_{,k}) = 0, \tag{1.14}$$

where

$$R = g^{ij}R_{ij}.^{8} \tag{1.15}$$

For the present case, since $g_{ij} = 0$, when i, j and k are different equation (1.14) is satisfied because of (1.3). For $j = i$ equation (1.14) is

$$R_{ii,k} - R_{ik,i} - \tfrac{1}{4}g_{ii}R_{,k} = 0,$$

that is

$$\sum_j \frac{1}{g_{jj}}R_{jiij,k} - \frac{1}{g_{jj}}R_{ijjk,i} - \frac{1}{4}g_{ii}\sum_{j,l}\frac{1}{g_{jj}}\cdot\frac{1}{g_{ll}}R_{jllj,k} = 0, \tag{1.16}$$

which is equivalent to

$$g_{kk}R_{jiij,k} + g_{jj}R_{kiik,k} - g_{ii}R_{kjjk,k} - 2\,g_{kk}R_{ijjk,i} = 0. \tag{1.17}$$

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[7] S., p. 294. The variable x_1 in (1.10) is a suitable function of x_1 used in the previous paper.

[8] R. G., p. 92, equation (28.19)

We have[9]

$$
\begin{aligned}
4R_{jiij} &= H_i^2\left(2\frac{\partial^2 \log H_i^2}{\partial x_j^2} + \frac{\partial \log H_i^2}{\partial x_j}\frac{\partial}{\partial x_j}\log\frac{H_i^2}{H_j^2}\right) \\
&+ H_j^2\left(2\frac{\partial^2 \log H_j^2}{\partial x_i^2} + \frac{\partial \log H_j^2}{\partial x_i}\frac{\partial}{\partial x_i}\log\frac{H_j^2}{H_i^2}\right) + \frac{H_i^2 H_j^2}{H_k^2}\frac{\partial \log H_i^2}{\partial x_k}\frac{\partial \log H_j^2}{\partial x_k}\,.
\end{aligned}
\tag{1.18}
$$

If we put

$$
\begin{aligned}
A_{ij} &= \frac{1}{H_j^2}\left(2\frac{\partial^2 \log H_i^2}{\partial x_j^2} + \frac{\partial \log H_i^2}{\partial x_j}\frac{\partial}{\partial x_j}\log\frac{H_i^2}{H_j^2 H_k^2}\right), \qquad (i, j, k \neq)\,. \\
B &= \frac{\partial H_i^2}{\partial x_k}\frac{\partial H_j^2}{\partial x_k} + \frac{\partial H_j^2}{\partial x_i}\frac{\partial H_k^2}{\partial x_i} + \frac{\partial H_k^2}{\partial x_j}\frac{\partial H_i^2}{\partial x_j}\,,
\end{aligned}
\tag{1.19}
$$

we have from (1.18),

$$
4\,R_{jiij} = H_i^2 H_j^2\,(A_{ij} + A_{ji}) + \frac{B}{H_k^2}\,. \tag{1.20}
$$

Substituting in (1.17) we have

$$
\begin{aligned}
H_i^2 H_j^2 H_k^2 \Big[&\frac{\partial}{\partial x_k}(A_{ij} + A_{ji} + A_{ik} + A_{ki} - A_{jk} - A_{kj}) \\
&+ \frac{\partial \log H_i^2}{\partial x_k}(A_{ij} + A_{ji} - A_{jk} - A_{kj})\Big] \\
&+ \frac{\partial B}{\partial x_k} - B\frac{\partial}{\partial x_k}\log(H_i^2 H_j^2 H_k^2) = 0\,.
\end{aligned}
\tag{1.21}
$$

If equation (1.17) is added to the equation obtained by interchanging the indices i and j in (1.17), we obtain

$$
g_{kk}(R_{jiij,k} - R_{ijjk,i} - R_{jiik,j}) = 0\,.
$$

But the quantity in parentheses is equal to zero identically.[10] Hence the system (1.21) is equivalent to that obtained by subtracting from it the equation resulting when the indices i and j are interchanged, that is

$$
\begin{aligned}
&\frac{\partial}{\partial x_k}(A_{ik} + A_{ki} - A_{jk} - A_{kj}) + \frac{\partial \log H_i}{\partial x_k}(A_{ij} + A_{ji} - A_{jk} - A_{kj}) \\
&\qquad - \frac{\partial \log H_j}{\partial x_k}(A_{ij} + A_{ji} - A_{ik} - A_{ki}) = 0\,.
\end{aligned}
\tag{1.21a}
$$

Consequently there are three conditions.

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[9] R. G., p. 119.

[10] Cf. R. G., p. 82, equation (26.3).

From the form of equations (1.17) it is evident that among the solutions of (1.21a) are Stäckel systems for spaces of constant Riemannian curvature.

In addition to the conditions (1.8) for the determination of σ we have also[11]

$$\sigma_{,ii} = \sigma_{,i}^2 - R_{ii} + \tfrac{1}{2} H_i^2 \left(\tfrac{1}{2} R - \Delta_1\sigma\right), \tag{1.22}$$

where

$$\Delta_1\sigma = \sum_i \frac{1}{H_i^2}\left(\frac{\partial\sigma}{\partial x_i}\right)^2, \tag{1.23}$$

and the conditions of integrability of these two sets of equations are satisfied in consequence of (1.14). If we put

$$\sigma = -\log\lambda, \tag{1.24}$$

equation (1.22) becomes

$$\frac{\partial^2\lambda}{\partial x_i^2} - \frac{1}{2}\cdot\frac{\partial\lambda}{\partial x_i}\cdot\frac{\partial \log H_i^2}{\partial x_i} + \frac{1}{2}\frac{\partial\lambda}{\partial x_j}\cdot\frac{1}{H_j^2}\frac{\partial H_i^2}{\partial x_j} + \frac{1}{2}\frac{\partial\lambda}{\partial x_k}\cdot\frac{1}{H_k^2}\cdot\frac{\partial H_i^2}{\partial x_k} = \lambda[R_{ii} - \tfrac{1}{2} H_i^2\left(\tfrac{1}{2} R - \Delta_1\sigma\right)], \tag{1.25}$$

where

$$\Delta_1\sigma = \frac{1}{\lambda^2}\sum_i \frac{1}{H_i^2}\left(\frac{\partial\lambda}{\partial x_i}\right)^2, \tag{1.26}$$

and (1.8) becomes

$$\frac{\partial^2\lambda}{\partial x_i \partial x_j} = \frac{1}{2}\frac{\partial\lambda}{\partial x_i}\frac{\partial \log H_i^2}{\partial x_j} + \frac{1}{2}\frac{\partial\lambda}{\partial x_j}\frac{\partial \log H_j^2}{\partial x_i}. \tag{1.27}$$

Before determining the Stäckel forms for a conformally flat V_3, we observe that, if we have such a form and a factor λ, there exists a transformation to cartesian coordinates

$$x = \psi_1(x_1, x_2, x_3), \qquad y = \psi_2, \qquad z = \psi_3 \tag{1.28}$$

such that

$$\frac{1}{\lambda^2}(H_1^2\,dx_1^2 + H_2^2\,dx_2^2 + H_3^2\,dx_3^2) = dx^2 + dy^2 + dz^2.$$

in accordance with a theorem of Liouville,[12] if we put

$$x' = \frac{x-\alpha}{A\sum(x-\alpha)^2}, \qquad y = \frac{y-\beta}{A\sum(x-\alpha)^2}, \qquad z = \frac{z-\gamma}{A\sum(x-\alpha)^2}, \tag{1.29}$$

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[11] R. G., p. 92, equation (28.18).

[12] *Note au sujet de l'article précédent*, Journal de Mathématiques, vol. 12 (1847), p. 265; also R. G., p. 94, Ex. 16.

where A, α, β, γ are arbitrary constants, we obtain another factor λ_1, yielding a solution and when x, y, and z in (1.28) are replaced by the ψ's, we have the corresponding transformation of coordinates. We find that

$$\lambda_1 = \lambda \quad A \quad \sum (x - \alpha)^2 . \tag{1.30}$$

Furthermore this gives the most general determination of the factor λ.

2. **Type 1.** For the expressions (1.10), that is

$$H_1^2 = \frac{1}{\phi(x_1)}, \qquad H_2^2 = x_1^2, \qquad H_3^2 = \psi(x_1), \tag{2.1}$$

we have from (1.19)

$$\begin{aligned} &A_{12} = A_{32} = A_{13} = A_{23} = 0, \\ &A_{21} = \frac{2\phi}{x_1}\left(\frac{\phi'}{\phi} - \frac{\psi'}{\psi}\right), \qquad A_{31} = \phi\left[\frac{2\psi''}{\psi} - \frac{\psi'^2}{\psi^2} + \frac{\psi'}{\psi}\left(\frac{\phi'}{\phi} - \frac{2}{x_1}\right)\right]. \end{aligned} \tag{2.2}$$

The equations (1.21a) impose the single condition

$$\frac{d}{dx_1}(\phi D) + \frac{\phi}{\psi}\left(\frac{\psi'}{x_1}\right)' = 0, \tag{2.3}$$

where

$$D = \frac{\psi''}{\psi} - \frac{\psi'^2}{2\psi^2} + \frac{\phi'}{2\phi}\left(\frac{\psi'}{\psi} - \frac{2}{x_1}\right). \tag{2.4}$$

A solution of (2.3) is $\psi = x_1^2$ and ϕ arbitrary. If $\psi \neq x_1^2$ and we put

$$D = \frac{1}{2} \cdot \frac{\psi'}{\psi}\left(\frac{\psi'}{\psi} - \frac{2}{x_1}\right) + f(x_1), \tag{2.5}$$

and substitute in (2.3) we find that $f = \dfrac{c}{\phi\psi}$, where c is a constant. Then from (2.4) and (2.5) we have as a first integral of (2.3)

$$\phi\left(\frac{\psi''}{\psi} - \frac{\psi'^2}{\psi^2} + \frac{1}{x_1} \cdot \frac{\psi'}{\psi}\right) + \frac{\phi'}{2}\left(\frac{\psi'}{\psi} - \frac{2}{x_1}\right) = \frac{c}{\psi}. \tag{2.6}$$

Again if we put

$$D = \frac{1}{x_1}\left(\frac{\psi'}{\psi} - \frac{2}{x_1}\right) + g(x_1), \tag{2.7}$$

and substitute in (2.3), we find that $g = \dfrac{d}{\phi x_1^2}$, where d is a constant, and from (2.4) and (2.7) we have the first integral

$$\phi\left(\frac{\psi''}{\psi} - \frac{1}{2}\frac{\psi'^2}{\psi^2} - \frac{1}{x_1} \cdot \frac{\psi'}{\psi} + \frac{2}{x_1^2}\right) + \frac{\phi'}{2}\left(\frac{\psi'}{\psi} - \frac{2}{x_1}\right) = \frac{d}{x_1^2}. \tag{2.8}$$

Eliminating ψ'' from (2.6) and (2.8) we have

$$\phi\left(\frac{\psi'}{\psi}-\frac{2}{x_1}\right)^2=\frac{2d}{x_1^2}-\frac{2c}{\psi}. \tag{2.9}$$

This satisfies (2.6) and (2.8). In particular $\psi=x_1^2$ is a special case, where $c=d$.

By means of (2.2) and (2.8) we find from (1.20)

$$\frac{R_{2112}}{H_1^2H_2^2}=\frac{1}{2}\cdot\frac{\phi'}{x_1},\qquad \frac{R_{3113}}{H_1^2H_3^2}=\frac{1}{2}\left[\frac{\phi}{x_1}\left(\frac{\psi'}{\psi}-\frac{2}{x_1}\right)+\frac{\phi'}{x_1}+\frac{d}{x_1^2}\right],$$
$$\frac{R_{3223}}{H_2^2H_3^2}=\frac{\phi}{2x_1}\cdot\frac{\psi'}{\psi}. \tag{2.10}$$

From these we find

$$R_{11}=\frac{1}{2\phi}\left[\frac{\phi}{x_1}\left(\frac{\psi'}{\psi}-\frac{2}{x_1}\right)+\frac{2\phi'}{x_1}+\frac{d}{x_1^2}\right],\qquad R_{22}=\frac{1}{2}x_1\phi\left(\frac{\phi'}{\phi}+\frac{\psi'}{\psi}\right),$$
$$R_{33}=\frac{\psi}{2}\left[\frac{2\phi}{x_1}\left(\frac{\psi'}{\psi}-\frac{1}{x_1}\right)+\frac{\phi'}{x_1}+\frac{d}{x_1^2}\right],\quad R=2\cdot\frac{\phi}{x_1}\left(\frac{\psi'}{\psi}-\frac{1}{x_1}\right)+\frac{2\phi'}{x_1}+\frac{d}{x_1^2}. \tag{2.11}$$

The space of constant curvature $-k$ is given by

$$\phi=\psi=kx_1^2+\frac{d}{2},\qquad c=kd. \tag{2.12}$$

3. **Type 2.** There are two cases of (1.11) according as σ_2 or σ_3 is a constant or not. By a suitable choice of coordinates these two cases have the respective normal forms

$$H_1^2=1,\qquad H_2^2=\phi^2(x_1),\qquad H_3^2=\phi^2(x_1)\cdot\psi^2(x_2), \tag{3.1}$$

and

$$H_1^2=1,\quad H_2^2=X_2\cdot\phi^2(x_1)(x_2-x_3),\quad H_3^2=X_3\cdot\phi^2(x_1)\cdot(x_3-x_2), \tag{3.2}$$

where X_σ for $\sigma=2,3$ is a function of x_σ at most.

For the form (3.1) we have

$$A_{12}=A_{13}=A_{23}=0,\qquad A_{21}=A_{31}=4\left(\frac{\phi'}{\phi}\right)',\qquad A_{32}=\frac{4}{\phi^2}\cdot\frac{\psi''}{\psi}. \tag{3.3}$$

Equations (1.21a) impose the single condition

$$\psi''=a\psi, \tag{3.4}$$

where a is a constant, and there is no restriction on ϕ. In this case we have

$$\frac{R_{2112}}{H_1^2H_2^2}=\frac{R_{3113}}{H_1^2H_3^2}=\frac{\phi''}{\phi},\qquad \frac{R_{2332}}{H_2^2H_3^2}=\frac{1}{\phi^2}(\phi'^2+a),$$
$$R_{11}=\frac{2\phi''}{\phi},\qquad \frac{R_{22}}{H_2^2}=\frac{R_{33}}{H_3^2}=\frac{1}{\phi^2}(\phi\phi''+\phi'^2+a), \tag{3.5}$$
$$R=\frac{2}{\phi^2}(2\phi\phi''+\phi'^2+a).$$

For spaces of curvature $-k$, ϕ must satisfy

$$\phi'^2 = k\phi^2 - a. \tag{3.6}$$

For the form (3.2) we have

$$A_{12} = A_{13} = 0, \qquad A_{21} = A_{31} = 4\left(\frac{\phi'}{\phi}\right)'$$

$$A_{23} = \frac{1}{\phi^2(x_2 - x_3)^2}\left[\left(\frac{1}{X_3}\right)' + \frac{2}{X_3}\cdot\frac{1}{x_2 - x_3}\right],$$

$$A_{32} = \frac{1}{\phi^2(x_2 - x_3)^2}\left[\left(\frac{1}{X_2}\right)' + \frac{2}{X_2}\cdot\frac{1}{x_3 - x_2}\right].$$

Equations (1.21a) impose the two conditions

$$(x_2 - x_3)^2\cdot\left(\frac{1}{X_2}\right)'' + 4(x_3 - x_2)\left(\frac{1}{X_2}\right)' + 2(x_3 - x_2)\left(\frac{1}{X_3}\right)' + 6\left(\frac{1}{X_2} - \frac{1}{X_3}\right) = 0,$$

$$(x_3 - x_2)^2\left(\frac{1}{X_3}\right)'' + 4(x_2 - x_3)\left(\frac{1}{X_3}\right)' + 2(x_2 - x_3)\left(\frac{1}{X_2}\right)' + 6\left(\frac{1}{X_3} - \frac{1}{X_2}\right) = 0,$$

and there is no restriction upon ϕ. Differentiating the first twice with respect to x_2, we have that $\left(\frac{1}{X^2}\right)^{IV} = 0$. Then we find that the above conditions are equivalent to

$$\frac{1}{X_\sigma} = 4ax_\sigma^3 + a_1\, x_\sigma^2 + a_2\, x_\sigma + a_3, \qquad (\sigma = 2, 3). \tag{3.7}$$

In this case we have

$$\frac{R_{1221}}{H_1^2\cdot H_2^2} = \frac{R_{1331}}{H_1^2 H_3^2} = \frac{\phi''}{\phi}, \qquad \frac{R_{2332}}{H_2^2 H_3^2} = \frac{a}{\phi^2} + \left(\frac{\phi'}{\phi}\right)^2,$$

$$R_{11} = \frac{2\phi''}{\phi}, \quad \frac{R_{22}}{H_2^2} = \frac{R_{33}}{H_3^2} = \frac{\phi''}{\phi} + \frac{a}{\phi^2} + \left(\frac{\phi'}{\phi}\right)^2, \quad R = \frac{2}{\phi^2}(2\phi\phi'' + a + \phi'^2). \tag{3.8}$$

For the space to be of constant curvature $-k$ the function ϕ must satisfy (3.6).

4. **Type 3.** For the form (1.12) we have

$$A_{12} = A_{32} = 0,$$

$$A_{21}\cdot\phi_1 = A_{31}\cdot(\phi_1 + e\phi_3) = \frac{1}{\phi_1 + e\phi_3}\left[2\phi_1'' - \phi_1'^2\cdot\frac{3\phi_1 + e\phi_3}{\phi_1(\phi_1 + e\phi_3)}\right],$$

$$A_{13}\cdot e(\phi_1 + e\phi_3) = A_{23}\cdot\phi_3 = \frac{1}{\phi_1 + e\phi_3}\left[2\phi_3'' - \phi_3'^2\cdot\frac{\phi_1 + 3e\phi_3}{\phi_3(\phi_1 + e\phi_3)}\right].$$

Equations (1.21a) impose the single condition

$$\begin{aligned}(4.1)\qquad &\frac{1}{\phi_1}\left[2\phi_1''(\phi_1+e\phi_3)-\frac{\phi_1'^2}{\phi_1}(3\phi_1+e\phi_3)\right]\\ &-\frac{1}{\phi_3}\left[2\phi_3''(\phi_1+e\phi_3)-\frac{\phi_3'^2}{\phi_3}(\phi_1+e\phi_3)\right]=a(\phi_1+e\phi_3)^3,\end{aligned}$$

where a is a constant. Differentiating successively with respect to x_1 and x_3, we obtain the conditions

$$(4.2)\qquad \frac{2\phi_1''}{\phi_1}-\frac{\phi_1'^2}{\phi_1^2}=3a\phi_1^2+b\phi_1+2c,\qquad \frac{2\phi_3''}{\phi_3}-\frac{\phi_3'^2}{\phi_3^2}=-3a\phi_3^2+2be\phi_3+2d,$$

where b, c, d are constants. In order that (4.1) and (4.2) be consistent, we must have $d=-c$ and

$$(4.3)\qquad \phi_1'^2=a\phi_1^4+b\phi_1^3+2c\phi_1^2+h\phi_1,\quad \phi_3'^2=-a\phi_3^4+be\phi_3^3-2c\phi_3^2+he\phi_3.$$

In this case we have

$$\begin{aligned}(4.4)\qquad &\frac{4R_{1221}}{H_1^2H_2^2}=a(2\phi_1-e\phi_3)+b, &&\frac{4R_{3223}}{H_2^2H_3^2}=a(\phi_1-2e\phi_3)+b,\\ &\frac{4R_{3113}}{H_1^2H_3^2}=2a(\phi_1-e\phi_3)+b, &&\frac{R_{11}}{H_1^2}=a\left(\phi_1-\frac{3}{4}e\phi_3\right)+\frac{b}{2},\\ &\frac{R_{22}}{H_2^2}=\frac{3}{4}a\left(\phi_1-e\phi_3\right)+\frac{b}{2}, &&\frac{R_{33}}{H_3^2}=a\left(\frac{3}{4}\phi_1-e\phi_3\right)+\frac{b}{2},\end{aligned}$$

$$R=\frac{5}{2}a(\phi_1-e\phi_3)+\frac{3}{2}b;$$

when $a=0$ the space is of constant curvature.

5. **Type** 4. For the form (1.13) we have

$$A_{ij}=\frac{1}{(x_j-x_i)^2(x_j-x_k)}\left[\left(\frac{1}{X_j}\right)'-\frac{2}{X_j}\left(\frac{1}{x_j-x_i}+\frac{1}{x_j-x_k}\right)\right],\qquad (i,j,k\neq).$$

Equations (1.21a) impose the one condition

$$\begin{aligned}&(x_2-x_3)^2\left[\left(\frac{1}{X_1}\right)'-\frac{2}{X_1}\left(\frac{1}{x_1-x_2}+\frac{1}{x_1-x_3}\right)\right]\\ &\quad+(x_3-x_1)^2\left[\left(\frac{1}{X_2}\right)'-\frac{2}{X_2}\left(\frac{1}{x_2-x_1}+\frac{1}{x_2-x_3}\right)\right]\\ &\quad+(x_1-x_2)^2\cdot\left[\left(\frac{1}{X_3}\right)'-\frac{2}{X_3}\left(\frac{1}{x_3-x_1}+\frac{1}{x_3-x_2}\right)\right]\\ &\quad=a(x_1-x_2)^2\cdot(x_2-x_3)^2\cdot(x_3-x_1)^2.\end{aligned}$$

Dividing by $(x_1 - x_2)^2$ and differentiating successively with respect to x_1, we obtain $\left(\frac{1}{X_1}\right)^v = 120a$. From this result and the above equations we have

$$\frac{1}{X_i} = ax_i^5 + a_1 x_i^4 + a_2 x_i^3 + a_3 x_i^2 + a_4 x_i + a_5 . \tag{5.1}$$

In this case we have

$$\begin{aligned} &\frac{4R_{ijji}}{H_i^2 \cdot H_j^2} = a[2(x_i + x_j) + x_k] + a_1 , \qquad (i, j, k \neq) \\ &\frac{R_{ii}}{H_i^2} = \frac{a}{4}(4x_i + 3x_j + 3x_k) + \frac{a_1}{2}, \qquad R = \frac{5}{2}a(x_1 + x_2 + x_3) + \frac{3}{2}a_1 . \end{aligned} \tag{5.2}$$

When $a = 0$, the space is of constant curvature.

6. **When λ is a function of $x^2 + y^2 + z^2$.** When a euclidean space is referred to any orthogonal coordinate system x_i, we have

$$\sum \left(\frac{\partial x}{\partial x_i}\right)^2 = \bar{H}_i^2 , \qquad \sum \frac{\partial x}{\partial x_i} \cdot \frac{\partial x}{\partial x_j} = 0 , \qquad (i \neq j) , \tag{6.1}$$

where Σ indicates the sum of terms for the cartesian coordinates x, y, z. From equations (6.1) and those arising by differentiation we find that

$$\begin{aligned} \frac{\partial^2 x}{\partial x_i^2} &= \frac{\partial \log \bar{H}_i}{\partial x_i} \cdot \frac{\partial x}{\partial x_i} - \frac{1}{2} \cdot \frac{1}{\bar{H}_j^2} \cdot \frac{\partial \bar{H}_i^2}{\partial x_j} \cdot \frac{\partial x}{\partial x_j} - \frac{1}{2} \cdot \frac{1}{\bar{H}_k^2} \frac{\partial \bar{H}_i^2}{\partial x_k} \cdot \frac{\partial x}{\partial x_k} , \\ \frac{\partial^2 x}{\partial x_i \, \partial x_j} &= \frac{\partial \log \bar{H}_i}{\partial x_j} \cdot \frac{\partial x}{\partial x_i} + \frac{\partial \log \bar{H}_j}{\partial x_i} \cdot \frac{\partial x}{\partial x_j} , \qquad (i, j, k \neq) . \end{aligned} \tag{6.2}$$

We seek the conditions that the fundamental form of a conformally flat V_3 can be expressed in the form

$$\frac{dx^2 + dy^2 + dz^2}{f(r^2)} , \qquad r^2 = x^2 + y^2 + z^2 . \tag{6.3}$$

Substituting $\lambda = \frac{1}{\sqrt{f(r^2)}}$ in (1.27), we obtain

$$\begin{aligned} 2\left(\frac{f''}{f'} - \frac{3}{2} \cdot \frac{f'}{f}\right) \sum x \frac{\partial x}{\partial x_i} \cdot \sum x \frac{\partial x}{\partial x_j} + \sum x \frac{\partial^2 x}{\partial x_i \, \partial x_j} \\ = \sum x \frac{\partial x}{\partial x_i} \cdot \frac{\partial \log H_i}{\partial x_j} + \sum x \frac{\partial x}{\partial x_j} \frac{\partial \log H_j}{\partial x_i} . \end{aligned}$$

Making use of (6.2) where $\bar{H}_i^2 = H_i^2 f(r^2)$, this equation is reducible to

$$\left(\frac{f''}{f'} - \frac{1}{2} \cdot \frac{f'}{f}\right) \sum x \frac{\partial x}{\partial x_i} \cdot \sum x \frac{\partial x}{\partial x_j} = 0 , \qquad (i \neq j) . \tag{6.4}$$

From the form of this equation it follows that if λ involves more than one of the x_i's, we have

$$\frac{f''}{f'} - \frac{1}{2} \cdot \frac{f'}{f} = 0,$$

that is

$$f = (ar^2 + b)^2, \tag{6.5}$$

which is the case of spaces of constant curvature.[13]

We consider the other possibility when λ is a function of x_1. From (1.27) we have that H_1 is a function of x_1 alone. Hence the only possibilities are Types 1 and 2. From (1.25) we have

$$\begin{aligned}
\frac{\partial^2 \lambda}{\partial x_1^2} - \frac{1}{2} \cdot \frac{\partial \lambda}{\partial x_1} \cdot \frac{\partial \log H_1^2}{\partial x_1} &= \lambda \left[R_{11} - \frac{1}{2} H_1^2 \left(\frac{1}{2} R - \frac{1}{\lambda^2} \cdot \frac{1}{H_1^2} \left(\frac{\partial \lambda}{\partial x_1} \right)^2 \right) \right], \\
\frac{1}{2} \cdot \frac{\partial \lambda}{\partial x_1} \cdot \frac{1}{H_1^2} \cdot \frac{\partial H_2^2}{\partial x_1} &= \lambda \left[R_{22} - \frac{1}{2} H_2^2 \left(\frac{1}{2} R - \frac{1}{\lambda^2} \frac{1}{H_1^2} \left(\frac{\partial \lambda}{\partial x_1} \right)^2 \right) \right], \\
\frac{1}{2} \cdot \frac{\partial \lambda}{\partial x_1} \cdot \frac{1}{H_1^2} \cdot \frac{\partial H_3^2}{\partial x_1} &= \lambda \left[R_{33} - \frac{1}{2} \cdot H_3^2 \left(\frac{1}{2} R - \frac{1}{\lambda^2} \frac{1}{H_1^2} \left(\frac{\partial \lambda}{\partial x_1} \right)^2 \right) \right].
\end{aligned} \tag{6.6}$$

From the last two of these equations we have

$$\frac{1}{2} \cdot \frac{\partial \log \lambda}{\partial x_1} \cdot \frac{1}{H_1^2} \cdot \frac{\partial}{\partial x_1} \log \frac{H_2^2}{H_3^2} = \frac{R_{22}}{H_2^2} - \frac{R_{33}}{H_3^2}. \tag{6.7}$$

For Type 1 this is

$$\left(\frac{d \log \lambda}{dx_1} - \frac{1}{x_1} \right) \left(\frac{\psi'}{\psi} - \frac{2}{x_1} \right) = \frac{d}{\phi x_1^2}, \tag{6.8}$$

and the second of (6.6) reduces to

$$\left(\frac{d \log \lambda}{dx_1} - \frac{1}{x_1} \right)^2 = \frac{d}{2\phi x_1^2}. \tag{6.9}$$

When $\psi = x_1^2$ and $d = 0$, equation (6.8) is satisfied identically and from (6.9) we have $\lambda = x_1$, which value satisfies the first of (6.6) also. Hence for any function ϕ the form

$$\frac{dx_1^2}{\phi} + x_1^2 (dx_2^2 + dx_3^2)$$

is a Stäckel form and

$$\frac{dx_1^2}{x_1^2 \phi} + dx_2^2 + dx_3^2$$

[13] R. G., p. 85.

the corresponding form for the flat $\bar{V}_3$. The transformation to cartesian coordinates is

$$x = \int \frac{dx_1}{x_1 \sqrt{\phi}}, \qquad y = x_2, \qquad z = x_3.$$

In this case λ is a function of x alone and not of r^2.

When $\psi \neq x_1^2$, equations (6.8), (6.9) and (2.9) are consistent, provided that $c = 0$, and we have that $\lambda = \sqrt{\psi}$, which satisfies the first of (6.6), whatever be ϕ and d. From (6.9) it follows that d is positive. If we put

$$\begin{aligned} x &= \sqrt{\frac{2}{d}} \cdot \frac{x_1}{\sqrt{\psi}} \cos\left(\sqrt{\frac{d}{2}} \cdot x_2\right), \\ y &= \sqrt{\frac{2}{d}} \cdot \frac{x_1}{\sqrt{\psi}} \cdot \sin\left(\sqrt{\frac{d}{2}} \cdot x_2\right), \\ z &= x_3, \end{aligned} \tag{6.10}$$

it follows from (2.9) with $c = 0$ that

$$dx^2 + dy^2 + dz^2 = \frac{1}{\psi}\left(\frac{dx_1^2}{\phi} + x_1^2\, dx_2^2 + \psi\, dx_3^2\right). \tag{6.11}$$

In this case λ is a function of $x^2 + y^2$ and not of r^2.

For the form (3.1) equation (6.7) is satisfied, and from the second of (6.6) we have

$$\left(\frac{d \log \lambda}{dx_1} - \frac{\phi'}{\phi}\right)^2 + \frac{a}{\phi^2} = 0. \tag{6.12}$$

Consequently $a < 0$ and replacing it by $-a^2$, we have from (3.4) $\psi = \sin a\, x_2$. If we replace ϕ by $a\phi$, ax_2 by x_2 and ax_3 by x_3, we have the form

$$dx_1^2 + \phi^2(dx_2^2 + \sin^2 x_2\, dx_3^2) \tag{6.13}$$

and from (6.12)

$$\frac{d \log \lambda}{dx_1} = \frac{\phi'}{\phi} \pm \frac{1}{\phi}. \tag{6.14}$$

The transformation to cartesian coordinates is

$$x = \frac{\phi}{\lambda} \cdot \sin x_2 \cos x_3, \qquad y = \frac{\phi}{\lambda} \cdot \sin x_2 \sin x_3, \qquad z = \frac{\phi}{\lambda} \cdot \cos x_2, \tag{6.15}$$

from which we have

$$x^2 + y^2 + z^2 = \frac{\phi^2}{\lambda^2}. \tag{6.16}$$

If in (6.14) we replace λ by any function of ϕ, we have either of two equations

for the determination of ϕ, according to the sign before $\frac{1}{\phi}$, so that λ shall be such a function and then from (6.16) we get λ as a function of r^2. In particular, if we take $\phi = \frac{1}{2} x_1$, and take the minus sign in the right hand member of (6.14), a solution is $\lambda = \frac{2}{x_1}$, in which case $\lambda^2 = \frac{1}{r}$.

For the form (3.2) equation (6.7) is satisfied and the second of (6.6) gives

$$\left(\frac{d \log \lambda}{dx_1} - \frac{\phi_1'}{\phi_1}\right)^2 = \frac{-a}{\phi_1^2}.$$

From this it is seen that a is negative and without loss of generality we take $a = -1$, and have (6.14) and the form

$$dx_1^2 + \phi_1^2 \cdot (x_2 - x_3)\left(\frac{dx_2^2}{f(x_2)} - \frac{dx_3^2}{f(x_3)}\right), \qquad f(x_\sigma) = 4(\alpha_1 - x_\sigma)(\alpha_2 - x_\sigma)(\alpha_3 - x_\sigma). \tag{6.17}$$

For the coordinates to be real no two of the α's can be zero and we may assume that

$$\alpha_1 > x_2 > \alpha_2 > x_3 > \alpha_3.$$

In consequence of (6.14) the transformation to cartesian coordinates is

$$x = \frac{\phi_1}{\lambda}\sqrt{\frac{(\alpha_1 - x_2)(\alpha_1 - x_3)}{(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)}}, \qquad y = \frac{\phi_1}{\lambda}\sqrt{\frac{(\alpha_2 - x_2)(\alpha_2 - x_3)}{(\alpha_2 - \alpha_1)(\alpha_2 - \alpha_3)}}, \tag{6.18}$$

$$z = \frac{\phi_1}{\lambda}\sqrt{\frac{(\alpha_3 - x_2)(\alpha_3 - x_3)}{(\alpha_3 - \alpha_1)(\alpha_3 - \alpha_2)}}$$

from which we have (6.16). The discussion following (6.16) applies equally to this case.

7. Gronwall[14] has reduced the Schrödinger equation for the helium atom to the form

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} + \frac{1}{z}\cdot\frac{\partial\psi}{\partial z} + \frac{1}{r}\left(E + \frac{1}{\sqrt{2(r+y)}} + \frac{1}{\sqrt{2(r-y)}} - \frac{1}{4\sqrt{r-x}}\right)\psi = 0.$$

If we substitute in this equation

$$\psi = \frac{\theta}{r^{1/4}\cdot\sqrt{z}},$$

[14] *A special conformally euclidean space of three dimensions occurring in wave mechanics*, Annals of Math., (2), vol. 33 (1932), pp. 279–293.

we obtain

$$(7.1)\quad r^{\frac{1}{2}} \sum \frac{\partial}{\partial x}\left(\frac{1}{\sqrt{r}} \cdot \frac{\partial \theta}{\partial x}\right) + \left(E + \frac{1}{\sqrt{2(r+y)}} + \frac{1}{\sqrt{2(r-y)}} - \frac{1}{4\sqrt{r-x}} + \frac{1}{4}\cdot\frac{r}{z^2} - \frac{3}{16r}\right)\theta = 0.$$

Comparing the first term of this equation with the first term of (1.1), we see that it is the case for which $H_i^2 = \frac{1}{r}$ $(i = 1, 2, 3)$. From the results of §6 we see that this form admits the Stäckel forms (6.13) and (6.17) in which $\phi = \frac{1}{2} x_1$, and the transformations to cartesian coordinates are given by (6.15) and (6.18) in these respective cases with $\phi = \frac{1}{2} x_1$. However, in neither case is the coefficient of θ in (7.1) expressible in the form

$$E + \sum_i \frac{X_i}{H_i^2}$$

where X_i is a function of x_i alone. Consequently this equation cannot be integrated by separation of the variables.

8. For $n = 3$ the condition (1.3) reduces to the first of equations (1.9), in consequence of the expression for R_{ij} previously given.[15] Thus for a V_3 any Stäckel system satisfying (1.3) is one of the forms (1.10), (1.11), (1.12) and (1.13). Hence we have:

For a Schrödinger equation in three variables the possible forms of the coefficients H_i^2 are given by (1.10), (1.11), (1.12) *and* (1.13), *in order that it may be solved by separation of variables.*

In this paper and the preceding one limitations were imposed on the functions appearing in the expressions by the additional requirement that the space be conformal euclidean and euclidean respectively.

Note added in proof (December 13, 1934): In discussing the solutions of the problem of §6, the cases where λ is a function of x_2 or x_3 alone were not treated. For the former case there is a solution when in (3.1) $\varphi = 1$, $\psi = e^{2x_2}$; this solution is $\lambda = e^{x_2}$ and the transformation to cartesian coordinates is

$$x = e^{-x_2} \cos x_1,\ y = e^{-x_2} \sin x_1,\ z = x_3;\ \frac{1}{\lambda^2} = x^2 + y^2.$$

When in (2.1) we take $\varphi = 1 - x_1^2$, $\psi^2 = 1$, there is a solution $\lambda = e^{x_3}$ and the transformation to cartesian coordinates is

$$x = e^{-x_3} x_1 \cos x_2,\ y = e^{-x_3} x_1 \sin x_2,$$
$$z = e^{-x_3} \sqrt{1 - x_1^2};\ \frac{1}{\lambda^2} = x^2 + y^2 + z^2.$$

Other possible solutions are equivalent to those previously given.

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[15] S., p. 291.

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DISCRETE ABELIAN GROUPS AND THEIR CHARACTER GROUPS

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Introduction

1. In this paper we construct the character group C of an arbitrary, discrete, abelian group G. We then discuss the geometrical properties of C, and show that C may always be identified with a closed subgroup of a toral group (defined in §3). Next, we prove the theorem that every closed subgroup C of a toral group may be identified with the character group of some discrete, abelian group G and that G may, in turn, be identified with the group of all continuous characters of C. Finally we show how C and G may be formalized in a new way which seems quite suggestive of the structure of these groups.

The theorem on the dual relations between discrete groups G and closed subgroups C of toral groups is a partial extension of Pontrjagin's relation of duality[1] between *countable* abelian groups and their character groups. However, it is an essentially simpler theorem, which can be proved without the Weyl-Peter integration process for constructing the characters of C. It would be interesting to prove directly that every bicompact, topological abelian group may be identified with a closed subgroup of a toral group. We would then have an elementary proof of Pontrjagin's relation in its extended form.

Our general method of procedure is based on the possibility of describing the groups G and C by certain systems of congruences derived by one of the present authors in a previous paper.[2] As regards notation, since we shall be dealing exclusively with abelian groups, we shall represent the group operation according to which the elements of a group are combined by *addition*. Then every finite linear combination of elements with integer coefficients will have an obvious meaning and will represent a definite group element. When we are dealing with toral groups we shall see that certain infinite combinations will also have a definite meaning, and that the coefficients need not be restricted to integer values.

Discrete Groups

2. Let G be an arbitrary, discrete, abelian group. We recall that a *basis* of G is any set of generators e_i of G. These generators need not be finite in number—

[1] L. Pontrjagin: *"The theory of topologic commutative groups,"* these Annals, vol. 35 (1934), 361-388.

[2] J. W. Alexander: *"On the characters of discrete abelian groups,"* these Annals, vol. 35 (1934), 389-395; also reference there to Wiener & Paley.

nor countable, for that matter—though each element x of G must be expressible in terms of a finite number of them:

$$x = \sum \xi_i e_i \qquad (\textstyle\sum \text{ finite, } \xi_i \text{ integers}). \tag{2:1}$$

Furthermore, the generators need not be independent. In other words, there may be certain linear forms

$$\sum p_{ji} e_i = 0 \qquad (\textstyle\sum \text{ finite, } p_{ji} \text{ integers}) \tag{2:2}$$

representing the zero element of G without vanishing identically. A set of such forms is called a *fundamental set* provided the most general linear form representing the zero element of G is a finite linear combination of forms of the set, with integer coefficients. The group G always has a basis (for example, the set of *all* elements of G), and to every basis there corresponds at least one fundamental set (for example, the set of *all* linear forms representing the zero element). The group G is completely determined when we know a basis of G and any corresponding fundamental set.

The group G may always be identified with a free group F reduced modulo a suitable subgroup H of F, $G = F \pmod{H}$. For we may take F to be the free group consisting of all finite linear combinations of the base elements e_i with integer coefficients, where each linear combination is now to be regarded as a distinct element of F. The subgroup H of F will then consist of all elements of F representing the zero element of G. The subgroup H is also a free group, since every subgroup of a free group is a free group.[3]

3. A *character* y of G may be defined as any single valued, linear, homogeneous function of a variable element x of G, such that the values of the functions are real numbers, modulo 1.[4] The functions y combine, by addition, to form a group called the *character group* C of G.

Now, suppose we have a group G, a set of base elements e_i and a corresponding set of fundamental relations (2:2). Then, clearly, a character y of G is completely determined when we know its values η_i at the base elements e_i respectively. For y is a linear function; therefore, its value at a general element (2:1) is given, at once, by the formula

$$y(x) = \sum \xi_i \eta_i, \qquad (\text{mod } 1). \tag{3:1}$$

The numbers η_i are not completely arbitrary. Since the function y is homogeneous it must vanish at $x = 0$. Therefore, by the fundamental relations (2:2), the η_i's must satisfy the congruences

$$\sum p_{ji} \eta_i \equiv 0, \qquad (\text{mod } 1). \tag{3:2}$$

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[3] See §8 of the present paper, concluding paragraph.

[4] Cf. L. Pontrjagin, loc. cit. The characters of G correspond to the various mappings of G on the group of rotations of the unit circle.

Conversely, if the η_i's satisfy the congruences (2:2), then the function y in (3:1) must be a character. For, in this case, y is single valued, in addition to being linear and homogeneous.

We shall think of the character group C as immersed in a group T composed of *all* determinations of the variables η_i, modulo 1, whether or not the determinations correspond to solutions of the congruences (3:2). The group T admits of a simple geometrical interpretation. Suppose we represent the range of each variable η_i by the points of a circle c_i with the angular coordinates $\eta_i/2\pi$ respectively. Then, each element of T may be identified with a set of points, one on each of the circles c_i. In other words, the group manifold of T may be thought of as the *product* of all the circular spaces c_i. We shall call the space T a toral space of as many dimensions (finite, or transfinite) as there are circles c_i. If there are only two circles c_i the space T may, of course, be pictured by an ordinary torus.

If we identify the group G with the group F (mod H), the toral group T may, at once, be identified with the group of all characters of the free group F generated by the base elements e_i without relations. The group C is the subgroup of the toral group T determined by (3:2).

4. Following Pontrjagin, we shall now assign a topology to the character group C of G by defining the *neighborhoods* of the elements y of C. Since C is a group, it will be sufficient to define the neighborhoods of the zero element. The neighborhoods of a general element y may then be taken as the images of the neighborhoods of 0 under the translation of the group manifold carrying 0 into y. By definition, there will be one neighborhood of zero,

$$N(\nu;\, x_1,\, x_2,\, \cdots,\, x_n) \tag{4:1}$$

corresponding to every choice of a positive integer ν and of a finite set of elements $x_1, x_2, \cdots, x_n$ of the *original group* G (n, finite but unbounded). Moreover, this neighborhood will be composed of all characters y such that their values at the points x_i are numerically smaller than $1/\nu$, modulo 1,

$$|\, y(x_i)\,| < 1/\nu\,, \qquad \pmod 1 \tag{4:2}$$

$$(i = 1, 2, \cdots, n)\,.$$

We shall call the neighborhoods (4:1) and their images the *N-neighborhoods* of C to distinguish them from certain other neighborhoods, presently to be described. In terms of the N-neighborhoods, we shall define limit points, continuity, etc., in the customary manner.

The N-neighborhoods (4:1) are, of course, independent of the basis of G. However, if we fix on a particular basis we may simplify the system (4:1) somewhat by discarding all neighborhoods except the ones of the form

$$N(\mu;\, e_{i_1},\, e_{i_2},\, \cdots,\, e_{i_m}) \tag{4:3}$$

where the determining elements e_{i_s} are now elements of the basis and not arbitrary elements of G. To show that the subsystem (4:3) is equivalent to the complete system (4:1) we must show that every neighborhood (4:1) contains a neighborhood (4:3). Let us therefore consider an arbitrary neighborhood (4:1). Each determining element x_i of this neighborhood is a finite linear combination of base elements,

$$x_j = \sum \xi_{ji} e_i , \qquad (j = 1, 2, \cdots, n). \tag{4:4}$$

Consequently, we can construct a neighborhood (4:3) contained in (4:1) by choosing as the determining elements e_{i_s} of (4:3) the elements appearing in the various right-hand members of (4:4) and by making the positive integer μ sufficiently large (as large, for example, as the product of the integer ν in (4:1) multiplied by the sum of the numerical values of all the coefficients ξ_{ji} in (4:4)).

A neighborhood of the restricted type (4:3) has an obvious geometrical interpretation when we picture the group C as a subgroup of the toral group T (cf. §3). For the value of a character y at a base element e_i is η_i; consequently a neighborhood (4:3) merely consists of all characters such that their m coordinates η_{i_s} $(s = 1, 2, \cdots, m)$, are numerically smaller than $1/\mu$ modulo 1. In other words, it consists of all characters such that their m coordinates η_{i_s} belong to certain open arcs α_{i_s} on the circles c_{i_s} respectively. This suggests a second way of defining a system of neighborhoods to determine the topology of C (and, incidentally, also of T). If we pick the circles c_{i_s} corresponding to any finite set of coordinates η_{i_s} and choose an arbitrary open arc α_{i_s}, one on each of them, then the arcs α_{i_s} will determine a subspace

$$V(\alpha_{i_1}, \alpha_{i_2}, \cdots, \alpha_{i_m}) \tag{4:5}$$

of C (and of T) consisting of all elements of C (or of T) such that their m coordinates η_{i_s} are represented by points of the arcs α_{i_s} respectively. We shall call such a subspace a *V-neighborhood* of every element of C (or of T) contained within it. (Certain V-neighborhoods associated with C may, of course, be vacuous, since they may not contain any element of C.) The V-neighborhoods of a point y of C (or of T) are obviously the images of the V-neighborhoods of the point 0 under the translation of the group manifold carrying 0 into y. Furthermore, every N-neighborhood of type (4:3) is a V-neighborhood of 0 and every V-neighborhood of 0 contains an N-neighborhood of type (4:3). Therefore the systems formed by the N- and V-neighborhoods must be equivalent. It will be convenient to regard the entire group space C (or T) as the V-neighborhood corresponding to the case where the arcs α_{i_s} are zero in number. The V-neighborhoods depend, of course, on the basis of G, but the topology they determine does not, since it is the same as the one determined by the N-neighborhoods.

In terms of the V-neighborhoods, we may verify, by inspection, that the group manifolds of C and T are Hausdorff spaces. However, if the group G is uncountable, the spaces C and T will not fulfil the countability axioms.

We close the first part of the discussion with the following theorem:

The character group C of an arbitrary discrete, abelian group G may always be identified with a closed subgroup of a toral group.

Proof. The group C is the set of all solutions of the congruences (3:2). Now, the coordinates η_i of a variable element y of T are obviously continuous functions of y. Therefore the forms $\Sigma p_{ji}\eta_i$ appearing in the left-hand members of the congruences (3:2) are also continuous, since each involves a finite number of coordinates only. Therefore, since the forms $\Sigma p_{ji}\eta_i$ vanish at all the elements of C they must also vanish, by continuity, at the accumulation points of C in T. In other words, the accumulation points of C must belong to C; whence, C must be closed.

Subgroups of Toral Groups

5. Let us now start afresh and see what can be said about an arbitrary closed subgroup C of a toral group T. We shall assume that the topologies of C and T are the ones determined by the V-neighborhoods of §4.

LEMMA. *Every toral group T is bicompact.*

Proof. We recall that an element y is called a *complete accumulation point* of an infinite set of points S if every neighborhood of y contains a subset of S of the same cardinal power $\aleph$ as S itself. In a similar fashion we shall say that a V-neighborhood is *complete* if it contains a subset of S of the same cardinal power as S. The group T is *bicompact* if every infinite set of elements of T has a complete accumulation point y in T. We shall assume that the coordinates of a variable element of T can be well ordered and shall determine the successive coordinates η_i of the desired accumulation point y by transfinite induction.

A neighborhood

$$V(\alpha_{i_1}, \alpha_{i_2}, \cdots, \alpha_{i_n})$$

such that the arcs α_{i_s} are written in their normal order, will be said to be of *index* i_n if i_n is the subscript of the last arc symbol α_{i_s}. We can tell whether or not such a neighborhood is a neighborhood of y as soon as we know the first i_n coordinates η_i of y; therefore, our meaning will be clear if we speak, somewhat loosely, about a neighborhood of y of index i_n before we have explicitly determined the coordinates of y subsequent to the i_n^{th} one. For the purposes of the induction, the space T itself is to be regarded as a V-neighborhood of index 0, corresponding to the case where there are no arcs α_{i_s} at all.

Let us now proceed with the induction. In selecting the i^{th} coordinate η_i we shall assume: (i) that if η_i is not the first coordinate then every coordinate η_j ahead of η_i has already been determined, and (ii) that the coordinate η_j has been so determined that every neighborhood of y of index j is complete. The problem will then be to select η_i in such a way that every neighborhood of y of index i is complete.

We prove that a suitable determination of η_i exists by a *reductio ad absurdum*.

If it did not, then, corresponding to each determination of η_i, there would be a neighborhood

$$V(\cdots, \alpha_i) \tag{5:1}$$

of index i, consistent with the given determinations of the η_j's, $j < i$, and such that its intersection with the set S had a lower cardinal number than $\aleph$. Moreover, the neighborhoods (5:1), corresponding to the various determinations of η_i would each determine an arc α_i such that the combined arcs α_i would cover all points of the circle c_i. Hence, by the Borel-Lebesgue theorem, there would be a finite subset

$$V^{(s)}, \qquad (s = 1, 2, \cdots, k) \tag{5:2}$$

of the neighborhoods (5:1) such that the arcs α_i which they determined also covered the circle c_i. Now, suppose we amalgamated the neighborhoods $V^{(s)}$ to form a single region

$$R = V^{(1)} + V^{(2)} + \cdots + V^{(k)}.$$

Then, clearly, the cardinal number of the intersections of this region with S would be lower than $k \cdot \aleph$, that is to say, lower than $\aleph$. Therefore, the regions R could not contain a neighborhood of y of index smaller than i for, by the hypothesis of the induction, such a neighborhood would contain a subset of S of cardinal power $\aleph$. On the other hand, we could easily construct a neighborhood of y contained in R and of index smaller than i. For, corresponding to each neighborhood $V^{(s)}$ in (5:2), we could form a neighborhood $\bar{V}^{(s)}$ such that the determining arcs of $\bar{V}^{(s)}$ were the same as those of $V^{(s)}$ with the omission of the last one (the arc on c_i). The intersection

$$\bar{V} = \bar{V}^{(1)} \cdot \bar{V}^{(2)} \cdots \bar{V}^{(k)}$$

of the neighborhoods $\bar{V}^{(s)}$ would obviously be a neighborhood of y of index smaller than i. Moreover, it would obviously be contained in the region R, for a point $\bar{y}$ of $\bar{V}$ would belong to all the component neighborhoods $V^{(s)}$ of R such that their determining arcs α_i covered the point of c_i representing the i^{th} coordinate of $\bar{y}$. We have thus arrived at a contradiction.

In conclusion, we have the result that the coordinates η_i of y may all be determined in such a way that every neighborhood of y, of whatever index i, is complete. The point y is, therefore, a complete accumulation point.

COROLLARY. *Every closed subgroup C of a toral group T is bicompact.*

For every infinite set S in C has a complete accumulation point in T, and every accumulation point of C must be in C.

It is, perhaps, worth observing that in the proof of the lemma we did not make use of the fact that the c_i's were *circles* but only the fact that they were spaces over which the Borel-Lebesgue theorem could be applied; that is to say, that they were *bicompact* spaces. The same formal proof, therefore, leads to the following general theorem:

THEOREM. *Let T be the product of an arbitrary set of bicompact spaces c_i and let T be topologized by the set of all neighborhoods*

$$V(\alpha_{i_1}, \alpha_{i_2}, \cdots, \alpha_{i_n})$$

determined by picking any finite number of the component spaces c_i and selecting an arbitrary neighborhood α_i on each. Then the space T is also bicompact.

6. THEOREM. *Let C be an arbitrary subgroup of a toral group T.*[5] *Then every continuous character x of C may be expressed by a finite linear form in the coordinates of a variable element y of C, with integer coefficients*

$$x(y) = \sum \xi_i \eta_i, \qquad \text{(mod 1)} \tag{6:1}$$

$$(\textstyle\sum \text{ finite}, \ \xi_i \text{ integers}).$$

Conversely, every such form represents a continuous character of C.

The second part of the theorem is immediate. The individual coordinates η_i are continuous characters of C, and the sum and difference of two continuous characters is a continuous character. Therefore, every finite form (6:1) must be a character. The coefficients ξ_i in (6:1) must, of course, be integers, otherwise the form would not be single valued, modulo 1.

In proving the first part of the theorem we shall begin with the simplest case of all, where the toral group T is of dimensionality one and can therefore be identified with the group of all rotations of a circle. Two subcases present themselves according as the subgroup C is, or is not, dense on the group T. If C is dense on T the domain of definition of a character x of C may at once be extended to T, by elementary continuity considerations. For the function x must be *uniformly* continuous on C, since it is both continuous and linear. Now, it is well known (and easily proved) that the continuous characters of the rotation group are all of the form

$$x(y) = \xi\eta, \tag{6:2}$$

where the coefficient ξ is an integer. Therefore the first subcase may be regarded as settled. If the group C is not dense on T it can only be a cyclic group of finite order k generated by an element $\eta = 1/k$. Therefore its characters x must again be of the form (6:1), where ξ/k denotes the value of x at the generating element $1/k$. The coefficient ξ must, of course, be an integer, since x must vanish at $\eta = 1 \equiv 0$, mod 1.

The case where T is of finite dimensionality n, $(n > 0)$, will be treated by induction. Let the coordinates of T be

$$\eta_i, \qquad (i = 1, 2, \cdots, n),$$

and let C' be the subgroup of C determined by requiring all coordinates but the last one to vanish. Then if x is any continuous character of C the value of x

[5] For this particular theorem we need not assume that the group C is closed in T.

over C' must be given by an expression of the form $\xi_n \eta_n$, by the one-dimensional case of the theorem which we have just disposed of. The character

$$x - \xi_n \eta_n \tag{6:3}$$

therefore vanishes at all elements of C whose first $n - 1$ coordinates vanish, whence it must be a single-valued function of these $n - 1$ coordinates. It follows, at once, by the hypothesis of the induction, that the character (6:3) must be expressible as a linear form in the $n - 1$ variables η_i $(i = 1, 2, \cdots, n-1)$. By transposing the term $\xi_n \eta_n$ we obtain the desired expression for x.

The case where T is of infinite dimensionality reduces, at once, to the finite one. Let x be any continuous character of C. Then, by the continuity of x, there exists a neighborhood

$$V(\alpha_{i_1}, \alpha_{i_2}, \cdots, \alpha_{i_n})$$

of the zero element such that at every element of C in this neighborhood the numerical value of x is smaller than $1/2$, modulo 1,

$$| x | < 1/2, \qquad (\text{mod } 1). \tag{6:4}$$

Now, the neighborhood V implies certain restrictions on the n coordinates η_{i_s} $(s = 1, 2, \cdots, n)$, but not on the others. Moreover, at every element of C such that these n coordinates vanish,

$$\eta_{i_1} \equiv \eta_{i_2} \equiv \cdots \equiv \eta_{i_n} \equiv 0, \qquad (\text{mod } 1), \tag{6:5}$$

the character x must also vanish. For if Condition (6:5) is satisfied at an element y of C it is also satisfied at every integer multiple λy of C. Therefore, if x failed to vanish at y we would merely choose λ so as to make the numerical value of $x(\lambda y)$ greater than $1/2$,

$$| x(\lambda y) | \equiv | \lambda x(y) | > 1/2, \qquad (\text{mod } 1), \tag{6:4}$$

contrary to (6:4). But if x has a unique value when the n coordinates η_{i_s} satisfy Condition (6:5) it must be a single-valued function of these coordinates alone, so that we are back once more to the finite case.

COROLLARY. *The domain of definition of a continuous character x of C may always be enlarged so as to include the entire toral space T.*

This is obvious, since the form (6:1) has a meaning at every element of T.

7. THEOREM. *Every closed subgroup C of a toral group T may be identified with the group of all solutions of a suitable system of congruences*

$$\sum p_{ji} \eta_i \equiv 0, \qquad (\text{mod } 1), \tag{7:1}$$

where the marks η_i are the coordinates of a variable y of T.

Again, we begin with the case where the dimensionality of T is unity. Suppose the closed subgroup C of T is the group T itself. Then, of course, the system (7:1) is empty. On the other hand, suppose C is a *proper* closed sub-

group of T. Then C can only be a cyclic group of some finite order κ, in which case, the system (7:1) is composed of a single congruence

$$\kappa\eta_1 \equiv 0, \pmod 1.$$

We treat the case where T is of finite dimensionality n, $(n > 1)$, by induction. Let the coordinates of an element y of T be $\eta_1, \eta_2, \cdots, \eta_n$. Then the element y' goes over, *by projection*, into an element $\eta_1, \eta_2, \cdots, \eta_{n-1}$ of the space T^* determined by the first $n-1$ coordinates of y. Moreover, the subgroup C of T goes over, by the projection of its elements, into a subgroup C^* of T^*, which is readily seen to be closed in T^* since C is compact and closed in T. Now, by the hypothesis of the induction, the subgroup C^* is determined by a system of congruences

$$\sum_{i<n} p_{ji}^* \eta_i \equiv 0, \pmod 1 \tag{7:2}$$

in the first $n-1$ coordinates, all of which must, of course, be satisfied by the coordinates of an element of C. The problem is to complete the system (7:2) so as to obtain a set of congruences determining C.

There are two cases to be considered according as the closed, one-dimensional subgroup C' of C determined by setting the first $n-1$ coordinates equal to zero is infinite or finite. If C' is infinite it must be the one-dimensional rotation group consisting of all elements of the form

$$(0, 0, \cdots, \eta_n), \tag{7:3}$$

where the coordinate η_n can take on any value, modulo 1. In this case, the congruences (7:2), regarded as congruences in all n variables η_i completely determine C. For every solution of (7:2) in the first $n-1$ variables is the projection of at least one element y of C, and every element of T with this same projection must belong to C, since it can only differ from y by an element of C of the form (7:3). If the group C' is finite it must be a cyclic group of order κ composed of the κ elements

$$(0, 0, \cdots, l/\kappa), \tag{7:4}$$

where l ranges over the values $0, 1, 2, \cdots, \kappa - 1$. It follows, at once, by considerations of linearity, that the last coordinate η_n of an element y of C is a κ-valued linear function

$$\eta_n = L(\eta_1, \eta_2, \cdots, \eta_{n-1})$$

of the first $n-1$ coordinates, such that two determinations of η_n corresponding to the same determination of the other coordinates can only differ by a number of the form l/κ, modulo 1. Thus we see that the function $\kappa\eta_n$ is *single-valued*, as well as linear, in the first $n-1$ variables η_i and is, therefore a character of C^*. Moreover, the character $\kappa\eta_n$ must be expressible as a linear form in the first $n-1$ coordinates, so that we have a congruence of the form

$$\sum_{i<n} p_i\eta_i + \kappa\eta_n \equiv 0, \pmod 1. \tag{7:5}$$

The combined relations (7:2) and (7:5) evidently determine C in this case. For every solution of (7:2) determines the projection of exactly κ elements of C, and the n^{th} coordinates of these κ elements are given by the κ solutions of (7:5).

The case where the dimensionality of T is infinite reduces, without difficulty, to the finite case. If we pick any finite set of coordinates

$$\eta_{i_1}, \eta_{i_2}, \cdots, \eta_{i_n} \tag{7:6}$$

and call the space of these coordinates T^* we may project the subgroup C of T into a closed subgroup C^* of T^* determined by a certain set of congruences

$$\sum p^*_{j_i}\eta_i \equiv 0\,, \qquad (\text{mod } 1)\,. \tag{7:7}$$

We shall prove that the combined congruences (7:7), corresponding to all possible choices of the finite set of coordinates (7:6), completely determine the group C. It is, of course, obvious that every element y of C corresponds to a solution of the combined congruences (7:7); the only point that requires proof is that every solution of the combined congruences corresponds to an element of C. Let $\bar{y}$ be an element of T determined by such a solution. Then each neighborhood $V(\alpha_{i_1}, \alpha_{i_2}, \cdots, \alpha_{i_n})$ imposes restrictions on a finite number of coordinates η_i only. Let the restricted coordinates be the ones in (7:6) and let the equations associated with these coordinates be (7:7). Then $\bar{y}$ corresponds to a solution of (7:7) (since it corresponds to a solution of all combined systems of the type (7:7)), and this solution determines the projection of at least one element y of C. Moreover, the element y is obviously in the neighborhood $V(\alpha_{i_1}, \alpha_{i_2}, \cdots, \alpha_{i_n})$. Thus, we have proved that in every neighborhood V of $\bar{y}$ there lies at least one element y of C. The element $\bar{y}$ must therefore belong to C, since C is closed. This concludes the proof of the theorem.

Duality

8. The complete duality between the discrete groups G and the toral groups C is a consequence of the following two theorems.

THEOREM 1. *Let C be an arbitrary toral group and G the group of all continuous characters of C. Then C may be identified with the character group of G.*

In §7 we showed that the continuous characters of C may always be expressed as finite linear combinations of the coordinates η_i with integer coefficients. The coordinates η_i therefore form a basis of G, $e_i = \eta_i$. Now suppose we form the set of *all* finite linear relations satisfied by the coordinates η_i

$$\sum p_{ji}\eta_i \equiv 0\,, \qquad (\text{mod } 1)\,. \tag{8:1}$$

Then, of course, these relations determine a set of fundamental relations of G,

$$\sum p_{ji}e_i = 0\,. \tag{8:2}$$

Conversely, by §3, the character group of the group G determined by (8:2) is the toral group C determined by (8:1), which establishes the theorem.

THEOREM 2. *Let G be an arbitrary discrete abelian group and C its character group. Then G may be identified with the group of all continuous characters of C.*

In proving this theorem we must be on guard against a possible error. Let (8:2) be an arbitrary set of fundamental relations determining C. Then, by §3, the character group C of G is surely the group determined by the relations (8:1) associated with (8:2). However, we cannot reverse the process and assert that the group G determined by (8:2) is the group of *all* continuous characters of C *unless* we are sure that every finite linear relation in the η_i's satisfied by the coordinates of a variable element y of C is a finite linear combination of Relations (8:1). *A priori*, Relations (8:1) might imply other relations not expressible as finite linear combinations of the given ones, in which case, the group of all continuous characters of C would not be G.

To settle this last point, we write the fundamental relations (8:2) in *normal form* as follows. First we arrange the base elements e_i in a well ordered set, $e_1, e_2, \cdots, e_\omega, \cdots$. We then divide the base elements into two classes, by transfinite induction. The first element e_1 is assigned to the first class if it is different from zero, otherwise to the second class. The i^{th} element e_i is assigned to the first class if no finite multiple of e_i is expressible as a finite linear combination of elements preceding e_i (with integer coefficients of course), otherwise to the second class. Let the elements of the first class, arranged in their natural order, be e_i' and the elements of the second class e_i''. Then, corresponding to each element e_j'' of the second class we form a relation

$$\sum_s p_{js}' e_s' + \sum_{t \leqq j} p_{jt}'' e_t'' = 0\,, \qquad (p_{jj}'' \neq 0)\,, \tag{8:3}$$

expressing the fact that a multiple $p_{jj}'' e_j''$ of e_j'' is expressible in terms of elements of the first class and of elements of the second class preceding e_j''. Moreover, we choose each relation in (8:3) such that p_{jj}'' is as small as possible numerically and (merely for convenience) positive. Then, it may be shown by transfinite induction, that every finite relation among the base elements e_i is expressible as a finite linear combination of Relations (8:3), with integer coefficients.[6] In other words, Relations (8:3) form a fundamental set. Moreover, if we form the corresponding relations

$$\sum_s p_{js}' \eta_s' + \sum_{t \leqq j} p_{jt}'' \eta_t'' \equiv 0\,, \qquad \pmod 1 \tag{8:4}$$

determining C, we can also show, by a similar transfinite induction, that every relation

$$\sum_s p_s' \eta_s' + \sum_{t \leqq j} p_t'' \eta_t'' \equiv 0\,, \qquad \pmod 1, (\textstyle\sum \text{ finite}) \tag{8.5}$$

implied by Relations (8:4) is a finite linear combination of these relations, with integer coefficients. The argument is, briefly, as follows. Relations (8:4)

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[6] Cf., for example, J. W. Alexander, loc. cit., p. 390.

imply no relation among the variables η'_s alone. For if we choose the η'_s at random we may then solve Relations (8:4) successively for the η''_t. We may thus assume that there are non-vanishing terms in the second sum in (8:5), and that the last coefficient p''_j is different from zero. Now this last coefficient must be exactly divisible by the coefficient p''_{jj} of the j^{th} equation (8:4), otherwise we could obviously find a solution of (8:4) but *not* of (8:5), such that all the η''_t ahead of η''_j and all the η'_s were zero. Therefore, we are able to subtract from (8:5) a suitable multiple of the j^{th} equation (8:4) and obtain a new relation similar to (8:5) and also dependent on (8:4), such that the last non-vanishing term in the second sum is a variable η''_i preceding η''_j. Now, by the hypothesis of the induction, this new relation is a finite linear combination of Relations (8:4); therefore, Relation (8:5) must also be such a combination. This completes the argument and establishes the duality between G and C.

We may observe at this point that if we regard the individual left-hand members of (8:3) as elements of the free group F generated by all finite integral combinations of the e'_i, e''_i, then these elements are certainly linearly independent. For each of them involves one base mark which is not contained in any of the preceding members. Therefore the forms in (8.3) generate a free subgroup H of F.

9. The group G is fully determined by certain relations (8:3) among its base elements, and the character-group C of G by certain relations (8:4) among the coordinates η_i of its elements. We shall close the discussion by giving a useful alternative way of obtaining the groups G and C in which G is determined by relations among suitably chosen coordinates and C by relations among suitably chosen base elements. The new way may be regarded as the dual of the old.

Let us take the base elements e_i (e'_i and e''_i), appearing in (8:3), treat them as independent units, and form a group F^* consisting of all *finite* linear combinations of the elements e_i with *real* coefficients

$$x = \sum \xi'_i e'_i + \sum \xi''_j e''_j , \qquad (\textstyle\sum \text{ finite, } \xi \text{ real}) . \tag{9:1}$$

Then there is a subgroup F of F^* consisting of all elements (9:1) with *integer* coefficients, which is obviously determined by requiring the coordinates ξ_i to satisfy the congruences

$$\xi'_i \equiv 0 , \qquad \xi''_j \equiv 0 , \qquad (\text{mod } 1) . \tag{9:2}$$

Moreover, the group G is the reduction of the group F, $G = F \pmod H$, obtained by imposing the conditions

$$\sum p'_{js} e'_s + \sum_{t \leqq j} p''_{jt} e''_t = 0 \tag{9:3}$$

on the base elements of F. In other words, the group G may be obtained from the group F^* by imposing the joint conditions (9:2) and (9:3).

We deal with the group C in a similar fashion. Corresponding to each co-

ordinate η_i of a variable element of C we choose a unit f_i and form a group T^* composed of all linear combinations

$$y = \sum \eta_i' f_i' + \sum \eta_j'' f_j'' \qquad (\textstyle\sum \text{infinite}, \eta \text{ real}) \tag{9:4}$$

of the units f_i, both finite and *infinite*, with arbitrary *real* coefficients. The group T^* contains a subgroup U consisting of all forms (9:4) with *integer* coefficients. By reducing the group T^* modulo the group U, we obtain a toral group $T = T^* \pmod U$. The group T is the group determined by imposing the conditions

$$f_i' = 0, \qquad f_j'' = 0 \tag{9:5}$$

on the units f_i, with the understanding that Relations (9:5) imply all other relations expressible as finite or *infinite* linear combinations of the given ones with *integer* coefficients. The group C is, of course, the subgroup of T determined by imposing the relations

$$\sum p_{js}' \eta_s' + \sum_{t \leqq j} p_{jt}'' \eta_t'' \equiv 0, \qquad \pmod 1 \tag{9:6}$$

on the coefficients. It is, therefore, obtainable from the group T^* by imposing the joint conditions (9:5) and (9:6).

Let us now make a change of basis in the space F^*, determined by the equations

$$\begin{aligned} \bar{e}_i' &= e_i' \\ \bar{e}_j'' &= \sum p_{js}' e_s' + \sum_{t \leqq j} p_{jt}'' e_t''. \end{aligned} \tag{9:7}$$

To prove that Equations (9:7) actually do determine a change of basis we must, of course, show that the old base elements e_i may be expressed in terms of the new ones. Now, for the elements of the first kind we have, at once,

$$e_i' = \bar{e}_i'; \tag{9:8}$$

whence, in place of the second set of equations, we may write

$$\sum_{t \leqq j} p_{jt}'' e_t'' = \bar{e}_j'' - \sum p_{js}' \bar{e}_s'. \tag{9:9}$$

But, in (9:9), each mark e_j'' makes its appearance for the first time in the j^{th} equation, where its coefficient p_{jj}'' is different from zero. Therefore, by transfinite induction, we may solve successive equations (9:9) for the successive variables e_j'' and so obtain relations of the form

$$e_j'' = \sum q_{js}' \bar{e}_s' + \sum_{t \leqq j} q_{jt}'' \bar{e}_t'', \qquad (\textstyle\sum \text{finite}). \tag{9:10}$$

Relations (9:10), together with (9:8), determine the inverse of the transformation (9:7); therefore, (9:7) determines a change of basis of F^*. It should be

noticed, however, that, unless the coefficients p''_{jj} all happen to be unity, the coefficients q'_{js} and q''_{jt} in (9:10) will not be integers, so that (9:7) will not be a change of basis of the subgroup F of F^*.

The change of basis (9:7) induces the contragredient change of coordinates

$$\begin{aligned} \xi'_i &= \bar{\xi}'_i + \sum p'_{si}\bar{\xi}''_s \\ \xi''_j &= \sum_{t \geqq j} p''_{tj}\bar{\xi}''_t \end{aligned} \tag{9:11}$$

which inverts into

$$\begin{aligned} \bar{\xi}'_i &= \xi'_i + \sum q'_{si}\xi''_s \\ \bar{\xi}''_j &= \sum_{t \geqq j} q''_{tj}\xi''_t . \end{aligned} \tag{9:12}$$

If we apply the change of basis (9:7) and the change of coordinates (9:11) to Relations (9:2) and (9:3) respectively, the first are transformed into

$$\begin{aligned} \bar{\xi}'_i + \sum p'_{si}\bar{\xi}''_s &\equiv 0 \\ \sum_{t \geqq j} p''_{tj}\bar{\xi}''_t &\equiv 0 \end{aligned} \quad (\text{mod } 1), \tag{9:13}$$

the second into

$$\bar{e}''_j = 0 . \tag{9:14}$$

These relations must now be interpreted in the proper manner. First, we observe that each of the old variables ξ_i appears in a finite number of the equations (9:12) only, and each of the new variables $\bar{\xi}_i$ in a finite number of the equations (9:11) only. Therefore, to the solutions of (9:2) such that all but a finite number of the variables ξ_i are zero there correspond solutions of (9:12) of a similar sort in the variables $\bar{\xi}_i$, and vice versa. Thus *the group G may be identified with the set of all solutions of the congruences* (9:13) *such that all but a finite number of the variables $\bar{\xi}_i$ are zero. The variables $\bar{\xi}_i$ are, however, of two sorts. Those of the first kind $\bar{\xi}'_i$ may be arbitrary real numbers, but those of the second kind $\bar{\xi}''_j$ are real numbers reduced modulo* 1. This last statement follows from Relations (9:14) which imply that if in any solution of (9:13) we change the values of the variables $\bar{\xi}''_j$ of the second sort by integers, we obtain a new solution representing the same element of G as the old one. Relations (9:13) determining G are analogous to Relations (8:4) determining C.

In the space T^* we shall make the transformation of coordinates

$$\begin{aligned} \bar{\eta}'_i &= \eta'_i \\ \bar{\eta}''_j &\mp \sum p'_{js}\eta'_s + \sum_{t \leqq j} p''_{jt}\eta''_t \end{aligned} \tag{9:15}$$

cogredient to (9:7) and the change of basis

$$\begin{aligned} f'_i &= \bar{f}'_i + \sum p'_{si}\bar{f}''_s \\ f''_j &= \sum_{t \geqq j} p''_{tj}\bar{f}''_t \end{aligned} \tag{9:16}$$

induced by this transformation of coordinates. The determining relations (9:5) and (9:6) of C will thus be transformed into

$$\begin{aligned} \bar{f}'_i + \sum p'_{si} \bar{f}''_s &= 0 \\ \sum_{t \geqq j} p''_{tj} \bar{f}''_t &= 0 \end{aligned} \tag{9:17}$$

and

$$\eta''_i \equiv 0 \pmod 1 \tag{9:18}$$

respectively. Their interpretation may be made as follows. We form what may be called a "pseudo basis" of C[7] consisting of the elements $\bar{f}_i$, such that every element of C may be expressed as a finite or *infinite* linear combination

$$y = \sum \eta'_i \bar{f}'_i + \sum \eta''_j \bar{f}''_j \tag{9:19}$$

of the base elements $\bar{f}_i$ with real coefficients. Then Relations (9:18) may be disposed of, at once, by restricting the coefficients of the second type η''_j to integer values. *The group C is then determined by the set of all finite and infinite linear forms* (9:19) *in the elements of the pseudo basis, such that the coefficients η'_i are arbitrary real numbers and the coefficients η''_j arbitrary integers. Moreover, each form* (9:19) *determines an element of C, and two forms* (9:19) *determine the same element of C if and only if their difference is a finite or infinite linear combination of the left-hand members of* (9:17), *with integer coefficients.* Of course, infinite linear combinations of Relations (9:17) have a definite meaning, since each base mark $\bar{f}_i$ appears in a finite number of the relations only. Relations (9:17) determining C are analogous to Relations (8:3) determining G.

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[7] J. W. A. Loc. cit.

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COUNTABLE TORSION GROUPS[1]

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Introduction. It is well known that the countable discrete abelian groups and the compact metric abelian groups are intimtately related, and that an analysis of the one class of groups is at the same time an analysis of the other.[2] While much is known about these classes of groups, the fundamental problem of their classification has been solved so far only in the case that the discrete group is of zero rank, i.e. a "Torsion" group. The corresponding topologic groups are then zero-dimensional. The remarkable solution of this restricted problem was given (from the point of view of the discrete groups) by Ulm.[3] The multiplicity of "convergence" questions with which Ulm's analysis is hampered suggests that a simpler proof is desirable. We have constructed such a proof, working first from the point of view of the topologic groups. This attack seemed natural, since the structural relations of the space might be expected to absorb some of the convergence difficulties. It transpired that, properly approached, these are not very deep. We have therefore "transposed" our original paper into the language of discrete groups, and there the formulation and solution of the problem is even simpler. The reader should be apprized that the essential novelty of this paper lies in the treatment of the reduced primary groups, beginning with §6.[4] We shall give the detailed analysis, presupposing an acquaintance with *finite* abelian groups and, to a limited extent, with transfinite countable ordinals.

Notation. We shall use additive symbols, 0 representing the "identity" element,[5] $-x$ the "inverse" to the element x, $x + y$ the "combination" of x and y. If $x_1, x_2, \cdots, x_n, \cdots$ is any set of sets of elements, then $[x_1, x_2, \cdots]$ denotes the set of *all* elements which are finite linear combinations of elements of these sets. If P, Q are sets, $P \cdot Q$ denotes the set of elements common to P and Q. The symbol $\subset$ denotes set-theoretic inclusion.

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[1] Presented to the American Mathematical Society, in the form given here, Oct. 27, 1934.

[2] See L. Pontrjagin: "The theory of topologic commutative groups;" J. W. Alexander, "On the characters of discrete abelian groups," and "On the homology groups of abstract spaces," all in these Annals **35** (1934).

[3] H. Ulm: "Abzählbar unendliche Abelsche Gruppen," Math. Ann. **107** (1933). The proof is in terms of transinfinite matrices. I am advised by Prof. E. Noether that Ulm is now in possession of group-theoretic proofs.

[4] Compare Prüfer, "Untersuchungen über die Zerlegbarkeit ⋯," Math. Zeitschrift **17** (1923). The theorems of the first four sections are due to Prüfer.

[5] When there is no possibility of confusion we may also use 0 for the number zero.

1. **Torsion Groups.** A torsion group T is a discrete countable abelian group, every element of which is of finite order. Thus, if $t \subset T$, there exists an integer N, $Nt = 0$. The smallest such integer is the *order* of t. This order is a function of the elements of T and need not be bounded. The word "discrete" shall mean merely that *finite* linear combinations only of elements of T have any meaning.

DEFINITION. T is the *direct sum* of a set of subgroups $S_1, S_2, \cdots$ if

i) $T \subset [S_1, S_2, \cdots]$

ii) $S_i \cdot [S_1, \cdots, S_{i-1}, S_{i+1}, \cdots] = 0$, for every i.

We shall write this $T = S_1 \oplus S_2 \oplus \cdots$ or $T = \sum \oplus S_i$. In particular, T is the direct sum of two subgroups P and Q if $T \subset [P, Q]$ and $P \cdot Q = 0$.

The following theorems are trivial.

THEOREM. *In order that* $T = \sum \oplus S_i$ *it is necessary and sufficient that either:*

1.1 *each element* t *of* T *is uniquely represented by a linear combination*

$$t = s_1 + s_2 + \cdots + s_n, \qquad n = n(t),$$

of elements of the S_i, *or*

1.2 i) *for every* t *and some* $n = n(t)$, $t \subset [S_1, S_2, \cdots, S_n]$

ii) *for every* i *and any* $m > i$

$$S_i \cdot [S_1, S_2, \cdots, S_{i-1}, S_{i+1}, \cdots, S_m] = 0.$$

THEOREM. *If* $T' = \sum \oplus S_i'$ *and* $T'' = \sum \oplus S_i''$ *and if for each* j *there exists an isomorphism* I_j *of* S_j' *and* S_j'', *then* T' *and* T'' *are isomorphic.*[6]

This isomorphism is given by the pairing off of elements $x' = \sum_1^n x'$ and $x'' = \sum_1^n x_j''$ such that $x_j' \subset S_j'$, $x_j'' \subset S_j''$, and x_j' and x_j'' correspond under I_j.

2. **Primary Groups.** For a fixed prime p, let T_p denote the set of elements of T such that $t \subset T_p$ if and only if there is an $n = n(t)$, $p^n t = 0$. Now if $x, y \subset T_p$ every element of the *finite* group $[x, y] \subset T_p$. Therefore T_p is a group. We shall call it a *primary* group (associated with the prime p).

THEOREM. $T = \sum \oplus T_p$, *summed over distinct primes.*

If $x \subset T$, the *finite* group $[x]$ is the direct sum of primary groups, and each of these belongs to the corresponding T_p. If $x \subset [T_{p_1}, T_{p_2}, \cdots, T_{p_n}]$, no element of the finite group $[x]$ can have an order $\equiv 0(q)$ where $q \neq p_i$, $i = 1, 2, \cdots, n$. The theorem is now a trivial consequence of 1.2.

COROLLARY. *In order that two torsion groups* T' *and* T'' *be isomorphic, it is necessary and sufficient that* T_p' *and* T_p'' *be isomorphic for every prime* p.

The necessity follows from the fact that the *order* of an element is an invariant of isomorphism, the sufficiency from 1.3.

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[6] Isomorphism is used throughout in the sense of a strictly one-to-one correspondence preserving the group operations.

3. Let T_p denote a primary group, for a *fixed* prime p.

DEFINITION 1. An element t is said (Prüfer) to be of infinite height in a subgroup G provided that for every integer $m \geqq 0$ there exists at least one element $y \subset G$ such that $t = p^m y$.

It is trivial that the set of elements of infinite height in G form a group.

DEFINITION 2. A subgroup R of T_p will be called a root-group (associated with p) provided *every* element of R is of infinite height in R.

3.1 Let R_∞ denote the set of all elements x such that T_p contains some root group $R_x \supset x$. Of course, $R_\infty \supset R_x$.

LEMMA. *If R_x, R_y are root groups, $[R_x, R_y]$ is a root group.*

For if $z = \alpha + \beta$, $\alpha \subset R_x$, $\beta \subset R_y$, then for any integer m, $\alpha = p^m\alpha'$ and $\beta = p^m\beta'$, $\alpha' \subset R_x$ and $\beta' \subset R_y$. Therefore $z = p^m(\alpha' + \beta')$, $\alpha' + \beta' \subset [R_x, R_y]$.

THEOREM. *R_∞ is a root group. It is maximal in the strong sense that it contains every root group of T_p. It is a characteristic subgroup of T_p in the sense that its properties define it uniquely.*

That R_∞ is a root group is an immediate consequence of the lemma. That it is *maximal* is obvious. That it is therefore a characteristic subgroup follows from the observation that the concept of infinite height, and therefore of root group, is an invariant of isomorphism.

3.2 DEFINITION. If x is any element and G any subgroup of T_p, then the least integer $N \geqq 1$ such that $Nx \subset G$ will be called the order of x *mod* G. It is clear that N is the order of the coset $x + G$ in the difference group (*finite* and primary) $[x, G]$ mod G. Therefore $N = p^m$ for some $m \geqq 0$.

3.3 THEOREM. *Any root group R of T_p is a direct "summand" of T_p* i.e. *$T_p = R + \bar{H}$, where $\bar{H}$ is a subgroup of T_p isomorphic to the difference group T* mod *R.*

Let us enumerate the elements of T_p: $x_1', x_2', \cdots, x_n', \cdots$. Suppose $x_0, x_1, \cdots, x_{n-1}$, $n \geqq 1$, $x_0 = 0$, have been so chosen that

$$\text{i)} \quad R \cdot [x_0, x_1, \cdots, x_{n-1}] = 0$$

and

$$\text{ii)} \quad x_1', x_2', \cdots, x_{n-1}' \subset [R, x_0, x_1, \cdots, x_{n-1}].$$

Let p^m, $m \geqq 0$, be the order of x_n' mod $[R, x_0, \cdots, x_{n-1}]$. Then for some x and z, $p^m x_n' = x + z$, where $x \subset [x_0, x_1, \cdots, x_{n-1}]$ and $z \subset R$. Let $z = p^m z'$ where $z' \subset R$, *and* let $x_n = x_n' - z'$. Then $x_n' = x_n + z' \subset [R, x_0, x_1, \cdots, x_n]$. *Now* if there is any element $z'' \subset R \cdot [x_0, x_1, \cdots, x_n]$, we can write $z'' = Nx_n + y$, $y \subset [x_0, x_1, \cdots, x_{n-1}]$. But then $Nx_n = z'' - y \subset [R, x_0, x_1, \cdots, x_{n-1}]$ and $N = kp^m$ for some integer k. But $kp^m x_n \subset [R, x_0, x_1, \cdots, x_{n-1}]$. Therefore $z'' \subset [R, x_0, \cdots, x_{n-1}]$. It follows from our antecedent choice of $x_0, x_1, \cdots, x_{n-1}$, that $z'' = 0$, since z'' was supposed in R. Then by a simple induction we may select elements $x_0, x_1, x_2, \cdots, x_n, \cdots$ so that i) and ii) hold for every n. Let $\bar{H} = [x_0, x_1, x_2, \cdots]$. It is now quite clear that $T_p = R \oplus \bar{H}$.

This has the following consequences, the proofs of which are trivial.

COROLLARY 1. *There exists a subgroup H of T_p such that*

$$T_p = R_\infty \oplus H,$$

where R_∞ is the maximal root group of T_p.

3.4 DEFINITION. A primary group whose maximal root group is the zero element will be called a reduced primary group. It is obvious that every *finite* primary group is reduced.

COROLLARY 2. *In the decomposition in Corollary* 1, *the group H is a reduced primary group.*

4. **Primary Root Group.** DEFINITION. A root group is called simple if it contains no subgroups which are also root groups other than itself and the identity.

THEOREM. *If z is any element of a root group R, there is a simple root group R_z, $R \supset R_z \supset z$.*

Write $z = z_0$. Suppose z_{n-1}, $n \geqq 1$, defined for some n. Let z_n denote any fixed element of R, such that $pz_n = z_{n-1}$. We may suppose z_n defined for every n. Let $R_z = [z_0, z_1, z_2, \cdots]$. It is clear from the construction that every element of R_z is of the form $x = Nz_n$, for some choice of an integer N and for some $n \geqq 0$. Then it is clear that $x = Np^m z_{n+m}$ for every $m = 0, 1, 2, \cdots$, and therefore R_z is a root group. It contains $z_0 = z$. If R' is any proper subgroup of R_z there is at least one $x = Nz_n \notin R'$. Then $z_n \notin R'$. But then $z_{n+m} \notin R'$, $m = 1, 2, \cdots$. Therefore R' is a finite group and certainly not a root group unless it is the identity. We observe at this point, without proof, that it is characteristic of simple primary root groups that they are discrete infinite groups containing no proper infinite subgroups.

4.1 Let us suppose that the order of the element z, above, is p^k. Then the order of z_n is p^{n+k}. Then if $x = Nz_n$ is of order p, $pNz_n = 0$ and $pN \equiv 0(p^{n+k})$. Then there is an integer N', $x = N'p^{k-1}z$. If $N' \equiv 0(p)$, $x = 0$. Then it follows that the number of distinct elements of R_z of order p is exactly p.

4.2 THEOREM. *Two simple root groups R' and R'' are isomorphic.*

Let $z_0' \neq 0$ and $z_0'' \neq 0$ be two elements of R' and R'' respectively, and of order p. Let $z_1', z_2', \cdots$, and $z_1'', z_2'', \cdots$, be chosen, as in 4, in R' and R'' respectively. The correspondence $Nz_n' \leftrightarrow Nz_n''$ is obviously an isomorphism of $[z_0', z_1', \cdots]$ and $[z_0'', z_1'', \cdots]$. But these are root groups and certainly neither of them the identity. Then they coincide with R' and R'' respectively.[7]

COROLLARY. *A simple root group has exactly p elements of order p.*

4.3 THEOREM. *A root group R is the direct sum of simple root groups.*

Let us enumerate the elements of R: $z_1, z_2, \cdots$, where none of these is the identity. Let R_1^* be a simple root group in R, containing z_1, by 4. Then $R = R_1^* \oplus R'$, by 3.3. Now R' is a root group. For if $x \subset R'$ and m is any pos-

[7] A simple root group is isomorphic to the multiplicative group of the p^n th roots of unity, n unbounded.

itive integer, $x = p^m(x' + y')$, $x' \subset R_1^*$ and $y' \subset R'$. But then $x - p^m y' = p^m x'$. Therefore both sides are 0, since one side belongs to R' and the other to R_1^*. But then $x = p^m y'$, and R' is a root group. Suppose R_k^* and R^k constructed for $k = 1, 2, \cdots, n-1$, so that

$$4.4 \qquad R = R_1^* \oplus R_2^* \oplus \cdots \oplus R_{n-1}^* \oplus R^{n-1}.$$

Let z_k be the first element in our enumeration not in $[R_1^*, \cdots, R_{n-1}^*]$. Then $z_k = z + z_k'$, $z_k \subset [R_1^*, \cdots, R_{n-1}^*]$ and $z_k' \subset R^{n-1}$. It is clear by the argument above that R^{n-1} is a root group. Let R_n^* be a simple root group of R^{n-1} containing z_k'. Then $R^{n-1} = R_n^* + R^n$, where R^n is a root group. If we continue this construction as long as any elements remain we shall have, finally,

$$R = \sum \oplus R_n^*.$$

This is trivial if there are only a finite number of these groups. But if there are infinitely many we observe that 1) for every k, $z_k \subset [R_1^*, R_2^*, \cdots, R_k^*]$, and 2) $R_i^* \cdot [R_1^*, \cdots, R_{i-1}^*, R_{i+1}^*, \cdots, R_m^*] = 0$, for every m and i, $m > i$, in virtue of 4.4 with $n = m + 1$. Then our assertion above follows from 1.2 and the theorem is proved.

COROLLARY. *If N is the number of summands in* 4.3, *then the number of elements of R of order p is p^N if N is finite and $\aleph_0$ if N is infinite.*

This is obvious from 4.1, since R_z of that section and any R_n^* are isomorphic (4.2).

THEOREM. *In order that two root groups R' and R'' be isomorphic it is necessary and sufficient that $N(R') = N(R'')$, where each of these is the number of simple summands in arbitrary decompositions* 4.3.

The sufficiency is obvious from 1.3, the necessity from the Corollary.

5. **Reduced Primary Group.** Let H denote a primary group which, in the next and succeeding sections, we shall suppose *reduced* (3.4). Now write $H = H_0$. If[8] H_μ has been defined for every ordinal μ *less* than a fixed ordinal λ, $\lambda > 0$, let H_λ denote the set of elements of H which are of infinite height in *every* H_μ, $\mu < \lambda$. Since the totality of elements of H is countable there must exist a *first* ordinal of the first or second ordinal class (i.e. a "countable" ordinal) such that $H_\kappa = H_{\kappa+1}$. Then it is clear that $H_\kappa = H_\nu$, for any ordinal $\nu > \kappa$. We shall call κ the *ordinal character* of H.

THEOREM. *H_λ is a characteristic subgroup of H, for every λ.*

This is trivial for $\lambda = 0$. Suppose it true for every $\mu < \lambda$, where λ is some ordinal > 0. If $x, y \subset H_\lambda$ and μ is any fixed ordinal $< \lambda$, then x and y are of infinite height in H_μ. Consequently $x + y$ is of infinite height in H_μ, and since this is true for every $\mu < \lambda$, $x + y \subset H_\lambda$. Clearly $H_\lambda \supset 0$, and with x also $-x$. Then H_λ is a group. Now since H_λ is defined in terms of the possibility of

[8] Following Ulm, loc. cit., p. 795.

solving "linear equations" which relate to the H_μ, $\mu < \lambda$, and these may be supposed isomorphism-invariant, it is clear that H_λ too is characteristic.

Since $H_\kappa = H_{\kappa+1}$, *every* element is of infinite height and H_k is a root group. It will be quite obvious later that it is the maximal root group of H.

6. Now if we consider that H is, by assumption, *reduced*, we must have $H_\kappa = 0$ (3.4).

LEMMA. *If $z \neq 0$ is an element of H, there is an ordinal $\lambda = \lambda(z) < \kappa$ such that $z \subset H_\lambda$, $z \not\subset H_{\lambda+1}$. This ordinal will be called the type of z: $\tau(z) = \lambda$. It is an invariant.*

There is certainly a *first* ordinal $\lambda' \leqq \kappa$ such that $z \not\subset H_\kappa$, since $z \neq 0$. Now λ' must have a predecessor. For if it had not, then for every ordinal $\mu < \lambda'$ there would be a μ', $\mu < \mu' < \lambda'$. Since $z \subset H_{\mu'}$, it is of infinite height in H_μ. But since this holds for every $\mu < \lambda'$, $z \subset H_{\lambda'}$. The contradiction establishes that λ' is of the form $\lambda + 1$, and λ is the desired ordinal. That the type-function $\tau(z)$ is invariant is a consequence of the invariance of H_λ, $\lambda \leqq \kappa$.[9] In a finite (primary) group every element is of type zero.

6.1 DEFINITION. A finite set of elements $x_1, x_2, \cdots, x_n$ of type $\lambda < \kappa$ will be called a λ-*set* provided that a relation

$$\text{A:} \quad a_1x_1 + a_2x_2 + \cdots + a_nx_n = p^sx + x' \qquad x \subset H_\lambda, \qquad x' \subset H_{\lambda+1}$$

always implies that for each $i = 1, 2, \cdots, n$,

$$\text{B:} \qquad \text{either } a_i \equiv 0(p^s) \quad \text{or} \quad a_ix_i \subset H_{\lambda+1}.$$

It will be called a $(\lambda; N)$-set provided, further, that each of its elements is of order p^N mod $H_{\lambda+1}$ (3.2). Clearly, $N \geqq 1$. Since $H_{\lambda+1}$ is a characteristic group, *order mod* $H_{\lambda+1}$ is an invariant concept, and it is clear that λ-set and $(\lambda; N)$-set is an invariant.

LEMMA. *Any non-vacuous subset of a λ-set is a λ-set.* Trivial.

6.2 *The complete invariant system of H: the "signature," $\sigma(H)$.*

For each ordinal λ, $0 \leqq \lambda < \kappa$, and for each integer $N \geqq 1$, let $\sigma(\lambda; N)$ be defined as the least upper bound of the set of all those numbers for which there exists a $(\lambda; N)$-set containing that number of elements. It is clear that $\sigma(\lambda; N)$ is zero, a positive integer, or plus infinity. Finally, let $\sigma(H)$ denote the totality of the numbers $\sigma(\lambda; N)$, which we may suppose arranged in their natural order:

$$\sigma(H) = (\sigma(0; 1), \sigma(0; 2), \cdots, \sigma(\omega; 1), \cdots, \sigma(\lambda; n), \sigma(\lambda; N + 1), \cdots).$$

$\sigma(H)$ will be called the signature of H. It is obviously an invariant of H, since the $\sigma(\lambda; N)$ are.[10]

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[9] It is immaterial how we define $\tau(0)$, since the element 0 is invariably an exception to any general statement which does not exclude it.

[10] The significance of this invariant is brought out in §10; our *definition* of the invariant differs from Ulm's (loc. cit.).

6.3 LEMMA. *Let $x_1', x_2', \cdots, x_n'$ be a λ-set, and w_i be any element of H_λ. If the order mod $H_{\lambda+1}$ of $x_i = x_i' + pw_i$ is not greater than that of x_i' then $x_1, x_2, \cdots, x_n$ is also a λ-set. Further, x_i and x_i' are of the same order mod $H_{\lambda+1}$.* Two λ-sets related in this way will be called equivalent.[11]

For if the x_i admit any relation A (in 6.1) where not all the a_i satisfy B, we may neglect all terms a_ix_i which belong to $H_{\lambda+1}$. Now there will be an integer $j \geqq 0$, such that all the remaining $a_i \equiv 0(p^j)$ but at least one is $\not\equiv 0(p^{j+1})$. Then if we make the substitution $x_i = x_i' + pw_i$ and transpose all terms to the right whose coefficient is $\equiv 0(p^{j+1})$, we shall have on the left terms involving the x_i' only, and there will be at least one. On the right we shall have an element of the form $p^{j+1}z$, $z \subset H_\lambda$. This contradicts the assumption that $x_1', \cdots, x_n'$ is a λ-system. Now, $p^{N_i-1}x_i = p^{N_i-1}x_i' + p^{N_i}w_i$. Then $p^{N_i-1}x_i \notin H_{\lambda+1}$, since x_i' is a λ-set. Therefore the order of x_i mod $H_{\lambda+1}$ is not less than p^{N_i} and, by assumption, not greater.

Now, for the next Corollary, let us observe that if x' is any element of type λ, and of order p^N mod $H_{\lambda+1}$ there must exist an element $w \subset H_\lambda$ such that $x = x' + pw$ is of order p^N. *For* if $p^Nx' = 0$, let $w = 0$. If

$$p^Nx' = w', \qquad 0 \neq w' \subset H_{\lambda+1},$$

then w' is of infinite height in H_λ and permits a solution of $p^{N+1}y = w'$, $y \subset H_\lambda$. Then let $w = -y$. Let us call a λ-set in which the order of each element is precisely its order mod $H_{\lambda+1}$, a *canonical λ-set*. Then we have proved the following

COROLLARY. *Every λ-set is equivalent to a canonical λ-set.*

LEMMA. *If $x_1, \cdots, x_n$ is a canonical λ-set, then,*

$$[x_1, x_2, \cdots, x_n] = [x_1] \oplus [x_2] \oplus \cdots \oplus [x_n].$$

For if there is any relation $a_1x_1 + \cdots + a_nx_n = 0$, then since $0 = p^s0$ for every s, we must have $a_ix_i \subset H_{\lambda+1}$. But then $a_i \equiv 0(p^{N_i})$ where p^{N_i} is the order mod $H_{\lambda+1}$ of x_i. Then $a_ix_i = 0$.

PRINCIPAL LEMMA. *If $x_1', \cdots, x_n'$ is a λ-set of which precisely $k \leqq n$ terms are of order p^N mod $H_{\lambda+1}$ and if $y_0', y_1', \cdots, y_k'$ is a $(\lambda; N)$-set, then there exists an element x_0 of order p^N mod $H_{\lambda+1}$ such that $x_0, x_1, \cdots, x_n$ is a λ-set.*

Let us replace each of the given λ-sets by equivalent canonical λ-sets: $x_1, \cdots, x_n$ and $y_0, \cdots, y_k$. Let y denote an arbitrary one of the y_i. Now if $y, x_1, x_2, \cdots, x_n$ is not a λ-set, then there must exist an integer $s \geqq 0$ and an element $w \subset H_\lambda$ such that $ay = p^sw + x$, where $a \not\equiv 0(p^s)$, $\not\equiv 0(p^N)$, and where $x \subset [x_1, \cdots, x_n]$ Now since $p^Ny = 0$, we can multiply this relation by a suitable integer so that

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[11] This and the notion of "canonical λ-set" below are introduced for this section only, for economy of speech. It is trivial that our definition satisfies the formal requirements for "equivalence."

it takes the form: $p^{N-1}y = p^N\overline{w} + \bar{x}$. Then if no "value" y_i of y makes y, $x_1, \cdots, x_n$ a λ-set we must have the following set of relations:

$$R_i: \quad p^{N-1}y_i = p^N w_i + z_i,$$

$$i = 0, 1, \cdots, k, \quad w_i \subset H_\lambda, z_i \subset [x_1, x_2, \cdots, x_k].$$

Since precisely k of the x's are of order p^N we may suppose that they are $x_1, \cdots, x_k$. Now $[x_1, \cdots, x_n] = [x_1, \cdots, x_k] \oplus [x_{k+1}, \cdots, x_n]$. Let $z_i = z_i' + z_i''$ be the corresponding resolution. Now if we multiply the relation R_i by p we shall have: $pR_i \equiv p^{N+1}w_i + pz_i' + pz_i'' = 0$. But then $pz_i' = 0$, since $x_1, \cdots, x_n$ is a canonical λ-set. Now $[x_1, \cdots, x_k] = [x_1] \oplus \cdots \oplus [x_k]$. Therefore it cannot contain more than k linearly independent elements of order p and there must exist a relation

$$R: \qquad a_0z_0' + \cdots + a_kz_k' = 0, \qquad 0 \leqq a_i < p, \text{ at least one } a_i \not\equiv 0 .$$

But this gives us R^*: $p^{N-1}R\{y_i\} = p^N R\{w_i\} + R\{z_i''\}$, where $R\{\psi_i\}$ is the form which results from substituting ψ_i for z_i' in the relation above. But if we expand pz_i'' in terms of $x_{k+1}, \cdots, x_n$, we see from the form of $pR_i \equiv p^{N+1}w_i + pz_i'' = 0$ that its coefficients must be $\equiv 0(p^{N+1})$, and therefore the coefficients of the expansion of z_i'' must be $\equiv 0(p^N)$ since $[x_{k+1}, \cdots, x_n] = [x_{k+1}] \oplus \cdots \oplus [x_n]$. It follows at once that the relation R^* is impossible and *therefore* there must exist a y_m such that $y_m, x_1, \cdots, x_n$ is a λ-set. Now if we write x_0' for y_m, this λ-set is equivalent to $x_0', x_1', \cdots, x_n'$ and the lemma is proved.

6.4 If an element x is of type λ, it cannot be of infinite height in H_λ. Let us agree to say that x is of height s in a subgroup G for *every* s for which $p^s y = x$ is solvable by a $y \subset G$. Then if $\tau(x) = \lambda$ there is a largest integer $s = s(x)$ such that x is of that height in H_λ.

DEFINITION. A subgroup G of H is called *perfect* provided that for *every* ordinal λ and *every* integer s, every element of G which is of type λ and height s in H_λ, mod $H_{\lambda+1}$, is also of height s in $G \cdot H_\lambda$, mod $G \cdot H_{\lambda+1}$.

$$\text{i.e. } \textit{if } z \subset G, \tau(z) = \lambda, \qquad x \subset H_\lambda, x' \subset H_{\lambda+1}, \textit{then}$$

$$z = p^s x + x' \quad \text{implies } z = p^s g + g'$$

where $g \subset G \cdot H_\lambda$, $g' \subset G \cdot H_{\lambda+1}$ and, of course, either g or g' or both may be zero.

LEMMA. *If G is perfect, if $\bar{G} = G \cdot H_\lambda$ mod $G \cdot H_{\lambda+1}$ is finite, if $\bar{x}_1, \cdots, \bar{x}_n$ is any independent (coset) basis for G, and if x_i is any element of $\bar{x}_i$, then $x_1, \cdots, x_n$ is a λ-set.*

For if there is any relation A(6.1) satisfied by the x_i, it is clear that the left side of this relation belongs to H_λ. Now if it happens that $a_1x_n + \cdots + a_nx_n \subset H_{\lambda+1}$, the corresponding $a_1\bar{x}_1 + \cdots + a_n\bar{x}_n = 0$. Then in this case every $a_i\bar{x}_i = 0$, and every a_ix_i belongs to $H_{\lambda+1}$. If this is not the case then $a_1x_1 + \cdots + a_nx_n$

is of type λ and we can suppose without loss that (see 6.1) $x \subset G \cdot H_\lambda$ and that $x' \subset G \cdot H_{\lambda+1}$, since G is perfect by assumption. But then, in $\bar{G}$,

$$a_1\bar{x}_1 + \cdots + a_n\bar{x}_n = p^s\bar{x},$$

and the a_i *must* satisfy B.

REMARK. It is clear that the condition $z = p^s x + x'$, above, is equivalent to the following condition: $z = p^s w$, $w \subset H_\lambda$, since x' is of *infinite* height and therefore of height s, certainly, in H_λ. We *cannot* "get rid" of g' in the same way.

7. FUNDAMENTAL THEOREM (ULM). *In order that two reduced*[12] *groups X and Y be isomorphic,*[13] $X \leftrightarrow Y$, *it is necessary and sufficient that their signatures*[14] *be the same:* $\sigma(X) = \sigma(Y)$.

We have already noticed that the necessity is obvious from the definition of the signature.

LEMMA. *If X^k, Y^k are perfect subgroups of X, Y, and if there is an isomorphism I_k: $X^k \overset{I_k}{\longleftrightarrow} Y^k$ which preserves types,*[15] *then I_k carries λ-sets into λ-sets.*

For if $y_1, \cdots, y_n$ is a λ-set and the corresponding set $x_1, \cdots, x_n$ is not then the x_i must satisfy a relation A (6.1) whose coefficients do not satisfy B. Now consider the corresponding form $a_1y_1 + \cdots + a_ny_n$. This cannot belong to $Y_{\lambda+1}$. For if it did each a_iy_i would belong to $Y_{\lambda+1}$ since the y_i form a λ-set. But then each a_ix_i would belong to $X_{\lambda+1}$, since I_k preserves types. Then $a_1x_1 + \cdots + a_nx_n$ does not belong to $X_{\lambda+1}$, and is certainly of type λ. But now because X^k is perfect we may suppose that the elements x and x' in A belong to $X^k \cdot X_\lambda$ and $X^k \cdot X_{\lambda+1}$ respectively. Then if y and y' are the corresponding elements in Y^k, $a_1y_1 + \cdots + a_ny_n = p^s y + y'$, and the a_i *do* satisfy B because the elements are all of the necessary types, and the y_i form a λ-set.

Proof of the Sufficiency. Induction. We shall suppose that for some integer $n \geqq 1$ we have already constructed two sets of groups:

$$0 = X^0 \subset X^1 \subset \cdots \subset X^{n-1} \quad \text{and} \quad 0 = Y^0 \subset Y^1 \subset \cdots \subset Y^{n-1}$$

such that

1) X^{n-1} is a finite[16] *perfect* subgroup of X and Y^{n-1} is a finite perfect subgroup of Y.

2) for every $k < n$ there is an isomorphism I_k, $X^k \overset{I_k}{\longleftrightarrow} Y^k$ such that:

2.1 if $x \subset X^k$ and $y \subset Y^k$ are corresponding elements of I_k, then $\tau(x) = \tau(y)$[17]

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[12] Associated with a fixed prime p.

[13] Isomorphism $\equiv$ simple isomorphism, throughout.

[14] Defined as in 6.2.

[15] i.e. $X^k \supset x \overset{I_k}{\longleftrightarrow} y \subset Y^k$ *implies* $\tau(x) = \tau(y)$, the first being computed in X the second in Y.

[16] *Remark:* For purposes of a later Theorem we shall wish to dispense with this. It will save confusion here if we merely signalize every use of this condition and leave for § 8 the explanation of our purpose.

[17] Where the first ordinal is computed in X, the second in Y.

2.2 I_k is an extension of I_{k-1}, i.e. if $x \subset X^{k-1}$ and $y \subset Y^{k-1}$ correspond under I_{k-1}, then they correspond under I_k.[18]

Now if $X^{n-1} = X$ and $Y^{n-1} = Y$ there is nothing to prove. Then we may suppose that either there is an element of X not in X^{n-1} or of Y not in Y^{n-1}. Our procedure will be entirely symmetric and for definiteness let us say that there is an element $z \subset X'$ but $\not\subset X^{n-1}$.[19] Let us now construct X^n.

7.1 Since z is of finite order, there is an integer $M \geqq 1$ such that $p^{M-1}z = z' \not\subset X^{n-1}$ but $pz' = p^M z' \subset X^{n-1}$. Now let us consider the set of elements $\{z' + g\}$ where $g \subset X^{n-1}$.[20] Since no element of this set is 0 and the set is finite[21] there must exist at least one g^* such that for $z'' = z' + g^*$ we have

i) $\tau(z'') \geqq \tau(z' + g)$ for every g

ii) if $\tau(z' + g) = \tau(z'')$ for some g, then the height of z'' in X_λ, where $\lambda = \tau(z'')$, is not lower than the height of $z' + g$ in X_λ. Now z'' being of finite height in X_λ there must exist a least integer $N \geqq 1$ such that z'' is *not* of height N in X_λ. Let x denote any fixed element of X_λ such that $p^{N-1}x = z''$ and let $X^n = [X^{n-1}, x]$. Since $X^n \supset x$, it contains z'' and therefore $z' = p^{M-1}z$.[22]

7.2 Let us show that X^n is perfect.[23] It is trivial that $p^N x \subset X^{n-1}$. Therefore every element of X^n has the form $g + ax$,[24] $0 \leqq a < p^N$. What we must prove is that for *every* ordinal $\mu < \kappa$ (= ordinal character of X),

$$\text{R:} \qquad g + ax = p^s w \qquad (w \subset X_\mu,\ \tau(g + ax) = \mu)$$

implies that $g + ax$ is of height s in $X^n \cdot X_\lambda$, mod $X^n \cdot X_{\mu+1}$. This will certainly be true whenever $\mu < \lambda = \tau(x)$ or whenever $a = 0$ for then in either case $ax = 0 \bmod X^n \cdot X_{\mu+1}$ and we know that X^{n-1} *is* perfect, by the induction. It will hold, also, when $\mu = \lambda$ and $a \equiv 0(p^s)$. For in this case $g = p^s g' + g''$ ($g' \subset X_\lambda$, $g'' \subset X_{\lambda+1}$) by the induction, and $ax = p^s(a'x)$ for some integer $a' \geqq 0$, so that $g + ax = p^s(g' + a'x) + g''$. To conclude the argument we shall show that the relation R leads to contradiction in all other cases.

In all other cases let j be the largest integer such that $a \equiv 0(p^j)$. Since in any event $0 \leqq a < p^N$ and $a = 0$ is now excluded, $N - j - 1 \geqq 0$. Let us multiply R by an integer bp^{N-j-1}, $0 < b < p$, so chosen that $abp^{j-1} = p^{N-1}(1 + a'p)$. Now since $p^N x \subset X^{n-1}$ and $p^{N-1}x = z''$, R takes the form

$$\hat{\text{R}}: \qquad \hat{g} + z'' = p^{N+s-j-1}\,\hat{w}, \qquad \hat{w} \subset X\hat{\mu}.$$

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[18] It will be kept in mind that we supposed $X^{k-1} \subset X^k$, $Y^{k-1} \subset Y^k$.

[19] The a priori possibility that we cannot find such an element in X but *could* have found it in Y will evaporate with the conclusion of the argument.

[20] Throughout this and the next sections, $g, g', g'', \cdots$, will invariably denote elements of X^{n-1}.

[21] See Remark, footnote to §7.

[22] This will be very important for "convergence," §7.

[23] This is the only point at which the possibility that there are elements of different types "interferes," in an essential way, with the argument.

[24] The reader will remember that $g, g', \cdots$, always denote elements of X^{n-1}.

We see at once that $\mu > \lambda$ violates the choice i) above. But if $\mu = \lambda$, then $s - j - 1 \geqq 0$ and this violates the choice of N in accordance with ii). Therefore X^n is perfect.

7.3 Since X^{n-1} is perfect and contains $p^N x$ it follows that $p^N x = p^N g_0 + g''$ where $g_0 \subset X^{n-1} \cdot X_\lambda$ and $g'' \subset X^{n-1} \cdot X_{\lambda+1}$. Let $\bar{x} = x - g_0$, and let $x_1, \cdots, x_m$ be elements of X^{n-1} belonging to independent generating cosets of the difference group $X^{n-1} \cdot X_\lambda \bmod X^{n-1} \cdot X_{\lambda+1}$. Then $x_1, \cdots, x_m$ is a λ-set by the lemma of 6.4. Now it is easy to see that every element of X^n is expressible uniquely in the form $g + a\bar{x}$, $0 \leqq a < p^N$ because it is trivial that $p^{N-1}\bar{x} \notin X^{n-1}$. But now the argument of 7.2 shows that $x_1, \cdots, x_m, \bar{x}$ is a λ-set with p^N as the order of $\bar{x} \bmod X_{\lambda+1}$. This may be verified quite easily by replacing x by $\bar{x}$ in that argument.

7.4 Let $y_1, \cdots, y_m$ be the elements of Y^{n-1} corresponding to the x_i above. Then they form a λ-set by the Lemma of 7. Now since $\sigma(X) = \sigma(Y)$ it is clear from the principal lemma of 6.3 that there must exist in Y an element y of order $p^N \bmod Y_{\lambda+1}$ such that $y_1, \cdots, y_m, y$ is a λ-set. Let $y' = p^N y \subset Y_{\lambda+1}$ and let γ'' be the element of Y^{n-1} corresponding to g'' above. Then $\gamma'' \subset Y_{\lambda+1}$ also and there must exist an element $y' \subset Y_\lambda$ such that $p^{N+1}y' = \gamma'' - y''$. Let $\bar{y} = y + py'$. Then $p^N\bar{y} = \gamma.''$ Finally, let $Y^n = [Y^{n-1}, \bar{y}]$.

Let us verify that every element of Y^n is expressible uniquely in the form $\gamma + a\bar{y}$, $0 \leqq a < p^N, \gamma \subset Y^{n-1}$. For this we need to show that $\tilde{\gamma} = p^{n-1}\bar{y} \notin Y^{n-1}$. Suppose it does belong to Y^{n-1} and let $\tilde{g}$ be the element corresponding to it in X^{n-1}. We know that $y_1, \cdots, y_m, \bar{y}$ is a λ-set, by the first lemma of 6.2, where $\bar{y}$ is of order $p^N \bmod Y_{\lambda+1}$. Therefore $\tilde{\gamma}$ is of type λ, which implies that $\tilde{g}$ is of type λ. On the other hand it is clear that $p\tilde{g} \subset X^{n-1} \cdot X_{\lambda+1}$. It follows, by our choice of the x_i, that there must exist a relation

$$a_0\tilde{g} + a_1x_1 + \cdots + a_mx_m = 0, \qquad \bmod X^{n-1} \cdot X_{\lambda+1},$$

where $0 < a < p$. This implies the corresponding relation:

$$a_0\tilde{\gamma} + a_1y_1 + \cdots + a_my_m = 0, \qquad \bmod Y^{n-1} \cdot Y_{\lambda+1}.$$

But this is impossible because $y_1, \cdots, y_m, \bar{y}$ is a λ-set and the coefficient of $\bar{y}$ is $a_0p^{N-1} \not\equiv 0(p^N)$.

Now to show that Y^n is perfect we can follow faithfully the argument of 7.2. If we do this we will be led, in the last lines, to a relation of the form

$$\hat{\mathrm{R}}: \qquad \hat{\gamma} + p^{N-1}\bar{y} = p^{N+s-j-1}\hat{w}, \qquad \hat{w} \subset Y_\mu.$$

Here, $\mu \geqq \lambda$ contradicts the fact that $y_1, \cdots, y_m, \bar{y}$ is a λ-set in which y is of order $p^N \bmod Y_{\lambda+1}$, since we know that $\hat{\gamma}$ can be expanded, mod $Y_{\lambda+1}$, into a form in the y_i, $i = 1, \cdots, m$ (here $\hat{\gamma}$ is the analagous element to $\hat{g}$ of 7.2).

7.5 It is obvious how we shall construct the isomorphism I_n: if γ and g correspond under I_{n-1}, let $\gamma + ay$ correspond to $g + ax$ for every a, $0 \leqq a < p^N$. Now it is trivial that this correspondence is an isomorphism and that it "extends" I_{n-1}. That it preserves types is also obvious on the ground that the sum

of two elements of *different* types must be of the lower type, while the element $g + ax\ (\gamma + ay)$, is certainly of type λ whenever g, (γ), is.[25] Then our induction is completed. It is clear that this induction cannot exhaust one group before the other at any *finite* stage. If it breaks off at a finite stage, then both groups are isomorphic. We may suppose the groups not finite.

7.6 Let us now suppose, for the moment only,[26] that we have constructed the groups X^n and Y^n, $n = 1, 2, \cdots$, in some arbitrary way, and let $X^* = [X^1, X^2, \cdots,]$ $Y^* = [Y^1, Y^2, \cdots]$. Then X^* and Y^* are isomorphic. This is quite clear, for if $x \subset X^*$ there is a first X^n which contains it and we set x in correspondence with that y of $Y^n \subset Y^*$ which corrresponds to it under the isomorphism I_n. It is trivial that this gives an isomorphism. To complete the proof of the theorem we need merely show that the groups X^n, Y^n can be chosen in such manner that $X^* = X$ and $Y^* = Y$. It is this which we have termed the "convergence." It is quite easily effected.

7.7 Let us enumerate all the elements of X: $x_1, x_2, \cdots$, and of Y: $y_1, y_2, \cdots$. Now suppose X^{n-1} and Y^{n-1} constructed for some $n \geqq 1$. Then if n is odd, let the element z of 7.1 denote the first element in the fixed enumeration of X which does not belong to X^{n-1}. If n is even, let z denote the first element in the enumeration of Y which does not belong to Y^{n-1}.[27] In either case perform the construction of 7.1. Now do this for every n. The group X^* determined in this way coincides with X, the group Y^* with Y.

The argument is symmetric, let us show $X^* = X$. If it does not, there is a first element x_k in the enumeration of X which is not an element of X^*. Let n be odd and large enough so that $X^{n-1} \supset x_i$, $i < k$. Then *at this point* z in 7.1 is the element x_k. Now if p^M is the order of x_i mod X^{n-1}, the group X^n constructed in 7.1 will contain $p^{M-1}x_i$. This is the last remark of 7.1. But then it is clear that X^{n+2M} will contain x_i.[28] The contradiction completes the proof of the Fundamental Theorem.

8. THEOREM.[29] *If H is a reduced (primary) group, any automorphism of H_ν can be extended to an automorphism of H, where ν is any ordinal not greater than the ordinal character of H.*

For the extreme values of ν this is, of course, trivial.

To prove the theorem we prepare two "copies" X and Y of H, and show that an isomorphism $X_\nu \leftrightarrow Y_\nu$ can be extended to X and Y. The proof of this coincides with our proof of the Fundamental Theorem with this modification. In the induction of 7 take $X^0 = X_\nu$ and $Y^0 = Y_\nu$ and the isomorphism I_0 to be the given one above. The condition that X^{n-1}, Y^{n-1} are *finite* is replaced by the

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[25] This is clear from the arguments above.

[26] See 7.7.

[27] It is quite unessential that the argument of 7.1 is stated in terms of X and X^{n-1}.

[28] This is not the closest fit.

[29] This is not a consequence of Ulm's theorem, but it also follows with no difficulty from his method of proof.

condition that the difference groups X^{n-1} mod X_ν and Y^{n-1} mod Y_ν are finite. Now the entire argument follows as before if we can show how to find an element z'' (in 7.1) with properties i) and ii). But here, if $g_1, \cdots, g_m$ denotes any complete set of elements of X^{n-1} which are incongruent mod X_ν, *this* set is finite and for this *restricted* set an element z'' does exist: $z'' = z' + g^*$, g^* some g_i (see 7.1).

Then all that remains to prove is that z'' satisfies i) and ii) for every $g \subset X^{n-1}$. Now observe that $\tau(z' + g) < \nu$ for every g.[30] Otherwise $z' + g \subset X_\nu \subset X^{n-1}$ and $z' \subset X^{n-1}$ which is impossible. Therefore $\tau(z'') < \nu$. Then it follows that if $g \subset X_\nu$, $\tau(z'' + g) = \tau(z'')$. From this it is clear that z'' satisfies *i*) for every g. Now if $z' + g^* + g = p^k x'$, where $g \subset X_\nu$ and $x' \subset X_\lambda$, $\lambda = \tau(z'') < \nu$, it is clear that $z' + g^* = p^k x''$, where $x'' \subset X_\lambda$. For g is of infinite height in X_λ. But this contradicts our choice of the integer k for the element z'' of *finite* height in X_λ. This established, we may follow the remainder of the proof above without further change because hereafter we use only the fact that X^{n-1} mod X_ν is finite for every n.

9. **Existence Questions.** Suppose we are given a system σ of sequences $\sigma(\lambda)$, $\lambda = 0, 1, 2, \cdots, \omega \cdots$, where $\lambda < \kappa$ of the first or second ordinal class, and $\sigma(\lambda) = \sigma(\lambda; 1), \sigma(\lambda; 2), \cdots$, each $\sigma(\lambda; N)$ being zero, a positive integer, or plus infinity, subject to the following conditions:

i) for every $\lambda < \kappa$, at least one $\sigma(\lambda; N) \neq 0$

ii) for every λ, $\lambda + 1 < \kappa$, and any integer n there is an $N \geqq n$ such that $\sigma(\lambda; N) \neq 0$.

We shall construct a group H for which $\sigma(H) \equiv \sigma$. We shall obtain H as the difference group of two free groups, the first being the group of all finite linear forms in a countable sequence of marks, $x_1, x_2, \cdots, x_n, \cdots$, the second a well defined subgroup. We shall formalize H by an infinite "square" matrix. The n^{th} column of this matrix will stand under the mark x_n, and above this mark we shall write a symbol λ_n. On the diagonal below it we shall write a symbol p^{s_n}.

Now we can certainly choose ordinal numbers λ_n, and integers $s_n \geqq 1$ such that if we are given any ordinal $\lambda < \kappa$ and any integer $N \geqq 1$ then the number of base marks x_n whose $\lambda_n = \lambda$ and $s_n = N$ is exactly the number $\sigma(\lambda; N)$. This follows from the observation that if we regard each $\sigma(\lambda; N)$ as a set containing $\sigma(\lambda; N)$ elements, then the totality of elements in all these sets is countable. Now for the construction of the matrix we shall agree that all "coördinates" above the main diagonal will be zero.

Let us construct the first column. If $\lambda_1 = 0$ all "coördinates" b_{i1}, where $i > 1$ gives the row, shall be zero. If $\lambda_1 \neq 0$, then by condition ii) we can find a sequence of ordinals $\lambda_{n1}, \lambda_{n2}, \cdots$, with corresponding diagonal exponents $s_{n1} < s_{n2} < \cdots$ such that if λ_1 has no predecessor, then $\lambda_{n1} < \lambda_{n2} < \cdots$ and

[30] $g, \cdots, \subset X^{n-1}$.

these converge to λ_1 whereas if λ_1 is of the form $\lambda + 1$ then every $\lambda_{n_i} = \lambda$. Now take those coördinates b_{i1} to be 1 for which n is one of the numbers $n_1, n_3, \cdots, n_{2m+1}, \cdots$ and take all others to be zero. Now let us put some warning signal over the diagonal numbers which now have a 1 to the left of them to tell us that the remaining coördinates shall be zero. If we take into account only those x_n, $n > 1$, which have no warning signal below them, and if we read a formal signature from them, we see that this "signature" will have the properties i) and ii), possibly for a "lower" κ. Then it is obvious how we shall continue the construction. The group H will be the difference group by the subgroup of "relations" corresponding to rows of the matrix.

The proof (inductive) that $\sigma(H) \equiv \sigma$ is tedious without being difficult, and will be omitted. We may remark that the matrix which we obtain is formally simpler than the corresponding canonical form of Ulm. First, the rows and columns are simply ordered. Second, each row has *at most* two non-zero coordinates. The conditions i) and ii) above are easily shown to be necessary.

10. THEOREM (*Prüfer*). *A necessary and sufficient condition that a group H shall be the direct sum of finite cyclic groups is that $H_1 = 0$; i.e. every element of H is of finite height.*

The necessity follows from the fact that if $H = \sum \oplus [x_i]$, every element of H is a unique form $\sum_1^n a_i x_i$ where each a_i is reduced mod the order of x_i, and the height of such an element is precisely the highest power of p which will divide *all* the a_i.

The sufficiency follows from the construction in 9, since in this case all non-diagonal coördinates are automatically zero.

COROLLARY. *The number $\sigma(0, N)$ of H, $H_1 = 0$, is the number of groups of order p^N in any decomposition of H into a direct sum of finite cyclic groups.*

By the construction in 9.

COROLLARY. *For a general H, $H_1 \neq 0$, $\sigma(\lambda; N)$ is the number of groups of order p^N in a similar decomposition of the difference group H_λ mod $H_{\lambda+1}$.*

11. THEOREM. *A necessary and sufficient condition that a given group H be expressible as the direct sum of groups isomorphic to given groups $H^1, H^2, \cdots$, is that*

$$\sigma(\lambda; N) = \sum \sigma_i(\lambda; N)$$

where the left number is computed in H, and $\sigma_i(\lambda; N)$ is computed in H^i. It is understood that if one side is infinite the other must be also.

This is an easy consequence of the fact that if $H = H' \oplus H''$ then $H_\lambda = H'_\lambda \oplus H''_\lambda$ and $\bar{H}_\lambda = \bar{H}'_\lambda \oplus \bar{H}''_\lambda$ where these are the corresponding difference groups mod $H_{\lambda+1}$, $H'_{\lambda+1}$, and $H''_{\lambda+1}$ respectively.

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TAUBERIAN AND SUMMABILITY THEOREMS ON DIRICHLET'S SERIES

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I. Introduction

1. Let $l_1, l_2 \cdots$ be a strictly increasing sequence tending to infinity and let $l_1 \geqq 1$; when l_0 occurs in any formula it is to be regarded as zero. Let $\sum_1^\infty a_n$ be any infinite series and,

$$A^\kappa(\omega) = \sum_{l_n \leqq \omega} a_n(\omega - l_n)^\kappa$$

where $\omega > 0$ and $\kappa \geqq 0$; when $\kappa = 0$, $A^0(\omega)$ is usually denoted by $A(\omega)$. When $\kappa > 0$ it is easily proved that

$$A^\kappa(\omega) = \kappa \int_0^\omega A(t)(\omega - t)^{\kappa-1}\, dt,$$

which, if multiplied by $\dfrac{1}{\Gamma(\kappa + 1)}$ is the Liouville-Riemann integral of order κ, of the function $A(\omega)$. With this notation, the following theorems are known.

THEOREM (A).[1] *Let* $\dfrac{a_n}{l_n - l_{n-1}} = O(l_n^\alpha)$ *where* $\alpha + 1 \geqq 0$. *Let* $0 < r \leqq 1$ *and* $\beta > 0$. *Let* $A^r(\omega) = S\omega^r + o(\omega^\beta)$ *where* S *is a constant. Then*

$$A(\omega) - S = o\left\{\omega \frac{\alpha r + \beta}{r + 1}\right\} F$$

THEOREM (B).[2] *Let* $p > 1$. *Let the sequence* (a_n) *satisfy the condition*

$$(X_p) \qquad \sum_{\nu=1}^{n} |a_\nu|^p \left(\frac{l_\nu}{l_\nu - l_{\nu-1}}\right)^p (l_\nu - l_{\nu-1}) = O(l_n).$$

If $A'(\omega) = l\omega + o(\omega)$, *then* $A(\omega) = l + o(1)$, *as* $\omega \to \infty$.

[1] K. Ananda Rau, "On the convergence and summability of Dirichlet's series," Proc. Lond. Math. Soc. (2) Vol. 34, part 6, pp. 414–440. The theorem quoted is theorem 4 of this paper which will, hereafter, be referred to as (A). All the results contained in the present paper were obtained at the suggestion of Mr. K. Ananda Rau to generalise the theorems contained in his paper when (a_n) satisfies appropriate conditions of type (X_p).

[2] O. Szasz, "Über Dirichletsche Reihen an der Konvergengrenze," Atti Del Congr. Internationale Dei Math. Bologne, Tomo. III, pp. 269–276. The theorem quoted is Lemma 4 (pp. 274) and forms a step in the proof of another result. The proof given there assumes that $\dfrac{l_n}{l_{n-1}} \to 1$. The complete theorem of which theorem B forms a step in the proof, will be proved without any restriction on the sequence (l_n) in the course of the present paper (See Theorem 9).

2. Theorem B gives rise to certain remarks. $\frac{A'(\omega)}{\omega}$ is Riesz's typical mean $(l, 1)$ and the theorem states that the convergence of this mean along with condition (X_p) involves the convergence of the series $\sum a_n$. Theorem (A) states the same thing when $\alpha = -1$ and $\beta = r = 1$. If (a_n) satisfies the condition imposed upon it in theorem (A) with $\alpha = -1$, then it satisfies (X_p) for all $p \geqq 1$, but not conversely. So (X_p) forms a more general condition than the one in theorem (A) when $\alpha = -1$. The question now arises whether, when $\alpha + 1 > 0$ and (a_n) satisfies appropriate conditions of type (X_p), similar results could not be obtained for $(A(\omega))$ from the behaviour of $(A^r(\omega))$. The object of this paper is to obtain such results and apply them to prove theorems regarding the summability of a Dirichlet's series from the nature of the function represented by it, as is done in the classical theorems of Schnee and Landau.[3]

3. Section II contains a collection of theorems which are known and which are used in the course of this paper. Section III contains the main results of this paper. Section IV contains two parts. The first is devoted to prove the theorem mentioned in footnote 2. The second part contains theorems of Schnee-Landau type.

II. Some known theorems

4. THEOREM (C).[4] *Let* $0 < r \leqq 1$ *and* $0 \leqq \xi \leqq x$. *Put*

$$\Phi_r(x) = \frac{1}{\Gamma(r)} \int_0^x \phi(t)\ (x - t)^{r-1}\, dt\,,$$

and

$$g(x, \xi) = \frac{1}{\Gamma(r)} \int_0^{\xi} \phi(t)\ (x - t)^{r-1}\, dt\,.$$

Then there exists a τ *in* $0 \leqq \tau \leqq \xi$ *such that*

$$g(x, \xi) = \Phi_r(\tau)\,.$$

Therefore also,

$$g(x, \xi) = O\ [\operatorname*{Max}_{0 \leqq \tau \leqq \xi} |\ \Phi_r(\tau)\ |]\,.$$

THEOREM (D).[4] *Let* $r > \kappa > 0$ *and* $V(x)$ *and* $W(x)$ *be two positive non-decreasing functions defined for* $x \geqq 0$. *Let* $\phi(x) = O(V)$ *and* $\Phi_r(x) = o(W)$. *Then*

$$\Phi_\kappa(x) = o\left\{V^{1-\frac{\kappa}{r}}\, W^{\frac{\kappa}{r}}\right\}.$$

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[3] See (A), theorems 9 and 10, pp. 433–38.

[4] Marcel Riesz, "Sur un théorème de la moyenne et ses applications" Acta Lit Ac. Sc. Reg. Univ. Hungaricae (Secto. Sci. Math.) 1 (1923), pp. 114–126. It is to be noted that in the definition of $\Phi_r(x)$, the condition $0 < r \leqq 1$ is unnecessary. It is sufficient if $r > 0$.

THEOREM (E).[5] *Let $s = \sigma + it$. Let $\sum a_n l_n^{-s}$ be summable by typical means (of type l_n) of sufficiently high order for $\sigma > d$ and let the function $f(s)$ defined by this series be regular for $\sigma > \eta$ where $\eta < d$. Further let κ, κ' be two positive numbers such that $\kappa' < \kappa$ and suppose that for every $\epsilon > 0$,*

$$f(s) = O[\,|\,t\,|^{\kappa'}]$$

uniformly for $\sigma \geqq \eta + \epsilon$. Then $(f(s))$ is summable (l, κ) for $\sigma > \eta$.

THEOREM (F).[6] *Let $A^{\kappa}(\omega) = o(\omega^{\kappa+\beta})$ where $\beta > 0$. Then $\sum a_n l_n^{-\beta}$ is either summable (l, κ) or is never summable (l, r) however large r be, κ being greater than or equal to zero.*

THEOREM (G).[7] *Let $\alpha_\nu \geqq 0$, $\beta_\nu \geqq 0$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Then*

$$(\sum \alpha_\nu \beta_\nu) \leqq (\sum \alpha_\nu^p)^{\frac{1}{p}} (\sum \beta_\nu^q)^{\frac{1}{q}}.$$

III. Main theorems

5. THEOREM 1. *Let the sequence (a_n) satisfy the condition,*

$$(Y_p) \qquad \sum_{\nu=1}^{n} |\,a_\nu\,|^p \left(\frac{l_\nu}{l_\nu - l_{\nu-1}}\right)^p (l_\nu - l_{\nu-1}) = O[l_n^{p(\alpha+1)+1}]$$

where $p > 1$ and $\alpha + 1 + 1/p \geqq 0$.[8] Then (a_n) satisfies (Y_{p_0}) for all p_0 such that $1 \leqq p_0 \leqq p$. Therefore if (Y_p) is true for some $p \geqq 1$ and not true for all $p \geqq 1$, then there is a finite λ such that (Y_p) is true for $p < \lambda$ and (Y_p) does not hold for $p > \lambda$.

Proof. Let $\epsilon > 0$ be such that $p - \epsilon \geqq 1$. Then using theorem (G), with $\dfrac{p}{p-\epsilon}$ for p, we get,

$$\sum_{\nu=1}^{n} |\,a_\nu\,|^{p-\epsilon} \left(\frac{l_\nu}{l_\nu - l_{\nu-1}}\right)^{p-\epsilon} (l_\nu - l_{\nu-1})$$

$$= \sum_{1}^{n} |\,a_\nu\,|^{p-\epsilon} \left(\frac{l_\nu}{l_\nu - l_{\nu-1}}\right)^{p-\epsilon} (l_\nu - l_{\nu-1})^{\frac{p-\epsilon}{p}} \cdot (l_\nu - l_{\nu-1})^{\frac{\epsilon}{p}}$$

$$\leqq \left\{\sum_{1}^{n} |\,a_\nu\,|^p \left(\frac{l_\nu}{l_\nu - l_{\nu-1}}\right)^p (l_\nu - l_{\nu-1})\right\}^{\frac{p-\epsilon}{p}} \left\{\sum_{1}^{n} (l_\nu - l_{\nu-1})\right\}^{\frac{\epsilon}{p}}$$

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[5] See Hardy and Riesz. "Dirichlet's series," Camb. Tracts No. 18, theorem 41, pp. 53–54.

[6] See (A), theorem 3, pp. 442. The assumption $\beta > 0$ may be replaced by $R(\beta) > 0$ where $R(\beta)$ is the real part of β.

[7] Known as Hölder's inequality.

[8] If (Y_p) holds for $\alpha + 1 + \dfrac{1}{p} < 0$, obviously all the a_n must be zero.

$$= O\left\{l_n^{[p(\alpha+1)+1]\frac{p-\epsilon}{p}+\frac{\epsilon}{p}}\right\}, \qquad \text{by } (Y_p)$$

$$= O\{l_n^{(p-\epsilon)(\alpha+1)+1}\}.$$

If $\alpha = -1$, (Y_p) reduces to (X_p).

THEOREM 2. *If (a_n) satisfies (Y_p) for a $p \geqq 1$, than $\sum a_n l_n^{-s}$ is absolutely convergent for $\sigma > \alpha + 1$, where $s = \sigma + it$, and $\alpha + 1 + \frac{1}{p} \geqq 0$.*

Proof. If. (Y_p) holds for a $p \geqq 1$ it holds for $p = 1$ by theorem 1. Therefore we have $\sigma_n = \sum_1^n | a_\nu | l_\nu = O(l_n^{\alpha+2}) \cdots (Y_1)$. Now

$$\sum_1^n | a_\nu | l_\nu^{-\sigma} = \sum_{\nu=1}^n | a_\nu | l_\nu \cdot l_\nu^{-(\sigma+1)}$$

$$= \sum_{\nu=1}^{n-1} \sigma_\nu \cdot \Delta l_\nu^{-(\sigma+1)} + \sigma_n l_n^{-(\sigma+1)}\,^{9}$$

$$= \sum_{\nu=1}^{n-1} \sigma_\nu \int_{l_\nu}^{l_{\nu+1}} \frac{d}{dx}(-x^{-\sigma-1})\,dx + \sigma_n l_n^{-(\sigma+1)}$$

$$= O\left\{\sum_{\nu=1}^{n-1} l_\nu^{\alpha+2} \int_{l_\nu}^{l_{\nu+1}} x^{-\sigma-2}\,dx + l_n^{\alpha-\sigma+1}\right\}, \qquad \text{by } (Y_1)$$

$$= O\left\{\int_{l_1}^{l_n} x^{\alpha+2} \cdot x^{-\sigma-2}\,dx + l_n^{\alpha-\sigma+1}\right\}, \qquad \text{since } \alpha + 2 > 0$$

$$= O\left\{\int_{l_1}^{l_n} x^{-(\sigma-\alpha)}\,dx + l_n^{-(\sigma-\alpha-1)}\right\}$$

$$= O(1)$$

provided $\sigma - \alpha > 1$ that is $\sigma > \alpha + 1$. Therefore the theorem is proved.

THEOREM 3. *Let $\dfrac{a_n}{l_n - l_{n-1}} = O(l_n^\alpha)$, $\alpha + 1 \geqq 0$. Then (a_n) satisfies (Y_p) for all $p \geqq 1$ and therefore this case may be considered as corresponding to $p = \infty$.*

Proof. We have,

$$\frac{| a_\nu |^p}{(l_\nu - l_{\nu-1})^p} = O(l_\nu^{\alpha p}).$$

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[9] $\Delta x_n = x_n - x_{n+1}$.

Therefore,

$$\sum_{\nu=1}^{n} |a_\nu|^p \left(\frac{l_\nu}{l_\nu - l_{\nu-1}}\right)^p (l_\nu - l_{\nu-1})$$

$$= O\left\{\sum_{\nu=1}^{n} l_\nu^{p(\alpha+1)} (l_\nu - l_{\nu-1})\right\}$$

$$= O\{l_n^{p(\alpha+1)+1}\}, \text{ since } \alpha + 1 \geqq 0.$$

Therefore the theorem is proved.

6. THEOREM 4.[10] *Let $\sum_1^\infty a_n$ be an infinite series. Let $0 < r \leqq 1$, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, so that $q > 1$. Let (a_n) satisfy (Y_p) in which $\alpha + 1 \geqq 0$. Let, $A^r(\omega) = S\omega^r + o(\omega^\beta)$ where $\beta > 0$ and S is a constant.*[11] *Then,*

$$A(\omega) - S = O\left\{\omega^{\frac{\left(\alpha+\frac{1}{p}\right)r+\beta/q}{r+1/q}}\right\}.$$

Proof. Case (i). Let $\beta > \alpha + r + 1$. Then,

$$\frac{\left(\alpha + \frac{1}{p}\right)r + \frac{\beta}{q}}{r + \frac{1}{q}} > \alpha + 1 \geqq 0. \tag{1}$$

Now, if (Y_p) holds, so does (Y_1). Therefore,

$$\sigma_n = \sum_1^n |a_\nu| \, l_\nu = O(l_n^{\alpha+2}). \tag{2}$$

Then,

$$A(\omega) = \left|\sum_{l_n \leqq \omega} a_n\right| \leqq \sum_1^n |a_\nu| \qquad \text{supposing } l_n \leqq \omega < l_{n+1}$$

$$\leqq \sum_1^n |a_\nu| \, l_\nu \cdot \frac{1}{l_\nu} = \sum_1^{n-1} \sigma_\nu \Delta \frac{1}{l_\nu} + \frac{\sigma_n}{l_n}$$

$$= O\left\{\sum_{\nu=1}^{n-1} l_\nu^{\alpha+2} \int_{l_\nu}^{l_{\nu+1}} \frac{dx}{x^2} + l_n^{\alpha+1}\right\}, \qquad \text{by (2)}$$

$$= O\left\{\int_{l_1}^{l_n} x^\alpha \, dx + l_n^{\alpha+1}\right\}, \qquad \text{since } \alpha + 2 > 0$$

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[10] See theorem (A) and footnote (1). The method of proof is the same as in (A).

[11] The constant S may be supposed zero without loss of generality. Otherwise we have merely to replace a_1 by $a_1 - s$. This will be supposed to have been done throughout the remainder of the paper even though in the statement S will be retained for form's sake.

$$= \begin{cases} O(l_n^{\alpha+1}), & \text{if } \alpha + 1 > 0 \\ O(\log l_n), & \text{if } \alpha + 1 = 0 \end{cases} \tag{3}$$

Therefore using (1), the theorem in this case, follows from (3) since $l_n \leqq \omega$.

Case (ii) $\beta \leqq \alpha + r + 1$. We suppose $S = 0$.[11] Let $\rho = \dfrac{\left(\alpha + \dfrac{1}{p}\right)r + \beta/q}{r + 1/q}$.

If the theorem is untrue, there is a $\delta > 0$ such that one of the inequalities (a) $A(\omega) \geqq \delta\omega^\rho$, (b) $A(\omega) \leqq -\delta\omega^\rho$ holds for an infinity of values of ω tending to infinity. We shall prove that either of these suppositions leads to a contradiction.

For example, let $A(\omega) \geqq \delta\omega^\rho$ for a sequence of $\omega \to \infty$. Let $\lambda = \dfrac{\beta - \alpha - 1/p}{r + 1/q}$ so that $\lambda \leqq 1$ since $\beta \leqq \alpha + r + 1$. Let m be such that $l_m \leqq \omega < l_{m+1}$. Let $x \geqq l_m$ and $l_{m+h} \leqq x < l_{m+h+1}$. Moreover let $x \leqq l_m + H\omega^\lambda$ where H is a constant which will be fixed later. Further let K denote an unspecified constant which is independent of m, x and ω but whose value need not be the same in the different places where it occurs below.

Now.[12]

$$|A(x) - A(\omega)| = |a_{m+1} + \cdots + a_{m+h}|$$

$$\leqq \sum_{\nu=1}^{h} |a_{m+\nu}| \left(\frac{l_{m+\nu}}{l_{m+\nu} - l_{m+\nu-1}}\right) (l_{m+\nu} - l_{m+\nu-1})^{\frac{1}{p}} \cdot \frac{(l_{m+\nu} - l_{m+\nu-1})^{\frac{1}{q}}}{l_{m+\nu}}$$

$$\leqq \left\{\sum_{\nu=1}^{h} |a_{m+\nu}|^p \left(\frac{l_{m+\nu}}{l_{m+\nu} - l_{m+\nu-1}}\right)^p (l_{m+\nu} - l_{m+\nu-1})\right\}^{\frac{1}{p}}$$

$$\times \left\{\sum_{\nu=1}^{h} \frac{l_{m+\nu} - l_{m+\nu-1}}{l_{m+\nu}^q}\right\}^{\frac{1}{q}}, \quad \text{by using theorem (G)}$$

$$\leqq K \frac{l_{m+h}^{\alpha+1+\frac{1}{p}}}{l_{m+1}} (x - l_m)^{\frac{1}{q}} \quad \text{in virtue of } (Y_p)$$

$$\leqq K\omega^{\alpha+\frac{1}{p}} (x - l_m)^{\frac{1}{q}}$$

since $l_m \leqq \omega < l_{m+1}$, $\alpha + 1 + \dfrac{1}{p} > \alpha + 1 \geqq 0$, and $\lambda \leqq 1$. The constant K above depends only on α, p and the constant involved in the O-sign of (Y_p) but can be supposed to be independent of H which will be chosen later to be at any rate $\leqq 1$. Therefore, for $l_m \leqq x \leqq l_m + H\omega^\lambda$,

$$A(x) \geqq \delta\omega^\rho - K\omega^{\alpha+\frac{1}{p}} (x - l_m)^{\frac{1}{q}}. \tag{4}$$

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[12] $A(x) - A(\omega) = 0$ if $l_m \leqq x < l_{m+1}$.

Therefore,[13]

$$\int_{l_m}^{l_m+H\omega^\lambda} A(x)\,(l_m + H\omega^\lambda - x)^{r-1}\,dx$$

$$\geqq \delta\omega^\rho \cdot \frac{(H\omega^\lambda)^r}{r} - K\omega^{\alpha+\frac{1}{p}}\,(H\omega^\lambda)^{r-1+\frac{1}{q}+1}$$

$$= \frac{H^r}{r}\,\omega^\beta\left(\delta - KrH^{\frac{1}{q}}\right) \tag{5}$$

using the values of λ and ρ.
Now, by hypothesis,

$$A^r(\omega) = r\int_o^\omega A(x)\,(\omega - x)^{r-1}\,dx = o(\omega^\beta)\,.$$

Therefore,

$$\int_o^{l_m+H\omega^\lambda} A(x)\,(l_m + H\omega^\lambda - x)^{r-1}\,dx = o\{(l_m + H\omega^\lambda)^\beta\}\,.$$

But $0 < r \leqq 1$ and therefore using theorem (C)[14] with $\xi = l_m$ we get,

$$\int_o^{l_m} A(x)\,(l_m + H\omega^\lambda - x)^{r-1}\,dx = o\{(l_m + H\omega^\lambda)^\beta\}\,.$$

Therefore,

$$\int_{l_m}^{l_m+H\omega^\lambda} A(x)\,(l_m + H\omega^\lambda - x)^{r-1}\,dx = o\{(l_m + H\omega^\lambda)^\beta\}$$

$$= o(\omega^\beta) \tag{6}$$

since $l_m \leqq \omega$, $\lambda \leqq 1$ and $\beta > 0$.
Comparing (5) and (6), we get,

$$\frac{H^r}{r}\left(\delta - KrH^{\frac{1}{q}}\right) = o(1)\,. \tag{7}$$

Now H is at our disposal. Choosing it so small that $\delta - KrH^{\frac{1}{q}}$ is positive and then fixing it, we see that according to (7), a positive constant independent of ω tends to zero as ω increases to infinity through a sequence of values. This is clearly impossible. Similarly the case $A(\omega) \leqq -\delta\omega^\rho$ for a sequence of $\omega \to \infty$ may be dealt with.

Therefore the theorem is proved.

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[13] We use the fact that $\int_a^b (x-a)^l\,(b-x)^m\,dx = (b-a)^{l+m+1}\,\frac{\Gamma(l+1)\,\Gamma(m+1)}{\Gamma(l+m+1)}$ provided $l+1>0$, $m+1>0$ and $b>a$.

[14] $A(x)$ and $A^r(x)$ are related to each other in the same manner as $\phi(x)$ and $\Phi_r(x)$ in theorem (c) except for constant factors depending on r only.

THEOREM 5. *Let* (a_n) *satisfy* (Y_p), $p > 1$ *and* $\alpha + 1 \geqq 0$. *Let* $A^r(\omega) = S\omega^r + o(\omega^r)$, $0 < r \leqq 1$. *Let* $0 \leqq \kappa < r$. *Then,*

$$\text{(i)} \quad A(\omega) - S = o\left\{\omega^{\frac{(\alpha+1)r}{r+1/q}}\right\}; \qquad \text{(ii)} \quad \frac{A^\kappa(\omega)}{\omega^\kappa} - S = o\left\{\omega^{\frac{(\alpha+1)(r-\kappa)}{r+1/q}}\right\}.$$

Proof. Let $\beta = r$ in theorem 4. Then we get the first result. Now let $0 < \kappa < r$. Since $\alpha + 1 \geqq 0$ the functions occurring in the o-signs of $A(\omega)$ and $A^r(\omega)$ are positive and non-decreasing. Therefore we can use theorem (D). Thus we get

$$A^\kappa(\omega) = o\left\{\omega^{\frac{(\alpha+1)r}{r+1/q}(1-\kappa/r)+r\cdot\kappa/r}\right\}$$

that is,

$$\frac{A^\kappa(\omega)}{\omega^\kappa} = o\left\{\omega^{\frac{(\alpha+1)(r-\kappa)}{r+1/q}}\right\}.$$

Here,[15] as usual, $S = 0$.

7. *Remark on theorems* 4 *and* 5. Theorem 4, and consequently theorem 5 also, has been proved under the assumption $p > 1$. If $p = 1$, the proof breaks down. If we look at the proof of the theorem 4 case (ii), we can easily verify that the inequality (4) and therefore (5) as well hold with $p = 1$ and $1/q = 0$. This may be seen from the evaluation of $A(\omega)$ in case (i) theorem 4. If we are to derive the same conclusion from these inequalities for the case $p = 1$ as when $p > 1$ we must be able to choose K so that $\delta - Kr$ should be positive. This is possible only if the o-sign in (Y_p) were replaced by the o-sign. Therefore, if this change be made in (Y_p) when $p = 1$, the theorems 4 and 5 remain true when $p = 1$ as well. This modification in the condition (Y_p) will be supposed to have been done when $p = 1$ occurs as a possibility in the hypothesis.

8. THEOREM 6.[16] *Let* (a_n) *satisfy* (Y_p) *for* $p \geqq 1$. *Let* $A^r(\omega) = S\omega^r + o(\omega^r)$ *where* r *is not restricted to be less than one. Let* $\epsilon > 0$ *be arbitrarily small and* $0 \leqq \kappa < r$. *Then,*

$$\text{(i)} \quad A(\omega) - S = o\left\{\omega^{\frac{(\alpha+1)r}{r+1/q}+\epsilon}\right\}; \qquad \text{(ii)} \quad \frac{A^\kappa(\omega)}{\omega^\kappa} - S = o\left\{\omega^{\frac{(\alpha+1)(r-\kappa)}{r+1/q}+\epsilon}\right\}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, *it being understood* $\frac{1}{q} = 0$ *if* $p = 1$.

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[15] In case $\alpha + 1 > 0$ the S may be omitted in the conclusion even if it be not zero in the hypothesis, since $S = o(\omega^\eta)$, if $\eta > 0$.

[16] This theorem corresponds to theorem (6) of (A). It may be noted that when (a_n) satisfies the condition of theorem (A) quoted at the beginning, (a_n) satisfies (Y_p) for all $p \geqq 1$ by theorem 3. Hence taking p large we can make $q \to 1$. But since ϵ is arbitrary we can actually replace q by *one* in this, and hence theorem 6 of (A) is seen to be a particular case of theorem 6 above.

Proof.[17] As usual $S = 0$. Let l and m be the lower limit of values of λ such that $A(\omega) = o(\omega^\lambda)$, $\frac{A'(\omega)}{\omega} = o(\omega^\lambda)$ respectively. Then we have, $A(\omega) = o(\omega^{l+\epsilon})$; also $A^r(\omega) = o(\omega^r)$ by hypothesis. Therefore by theorem (D), $m \leqq (l + \epsilon)\left(1 - \frac{1}{r}\right)$ and since ϵ is arbitrarily small, $m \leqq l\left(1 - \frac{1}{r}\right)$. Therefore

$$\frac{A'(\omega)}{\omega} = o\{\omega^{l(1-1/r)+\epsilon}\}$$

and so using theorem 4 with $r = 1$, $\beta = l\left(1 - \frac{1}{r}\right) + 1 + \epsilon$ we get since ϵ is small,

$$l \leqq \frac{\left(\alpha + \frac{1}{p}\right) + \underline{l\left(1 - \frac{1}{r}\right) + 1}}{1 + \frac{1}{q}};$$

that is

$$l\left(1 - \frac{\frac{1}{q} - \frac{1}{qr}}{1 + \frac{1}{q}}\right) \leqq \alpha + 1 \quad \text{or} \quad l \leqq \frac{(\alpha + 1)r}{r + \frac{1}{q}},$$

which proves the first relation as l is the lower limit of values of λ for which $A(\omega) = o(\omega^\lambda)$. The second follows by the use of theorem (D) as theorem 5 follows from theorem 4.

LEMMA 1. *Let* $b_n = a_n l_n^{-\gamma}$. *Let* (a_n) *satisfy* (Y_p) *for a* $p \geqq 1$. *Then* (b_n) *satisfies* (Y_p) *with* $\alpha - \gamma$ *in place of* α, *provided* $\alpha - \gamma + 1 + \frac{1}{p} > 0$ *as well as* $\alpha + 1 + \frac{1}{p} \geqq 0$.

Proof. By hypothesis,

$$H_n = \sum_{\nu=1}^{n} |a_\nu|^p \left(\frac{l_\nu}{l_\nu - l_{\nu-1}}\right)^p (l_\nu - l_{\nu-1}) = O\{l_n^{p(\alpha+1)+1}\}. \tag{8}$$

Therefore,

$$G_n = \sum_{\nu=1}^{n} |b_\nu|^p \left(\frac{l_\nu}{l_\nu - l_{\nu-1}}\right)^p (l_\nu - l_{\nu-1})$$

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[17] The method of proof is due to Mr. Bosanquet.

$$= \sum_{\nu=1}^{n} |a_\nu|^p \left(\frac{l_\nu}{l_\nu - l_{\nu-1}}\right)^p (l_\nu - l_{\nu-1})\, l_\nu^{-\gamma p}$$

$$= \sum_{\nu=1}^{n-1} H_\nu \Delta l_\nu^{-\gamma p} + H_n l_n^{-\gamma p}$$

$$= O\left\{\sum_{1}^{n-1} l_\nu^{p(\alpha+1)+1}\left[\int_{l_\nu}^{l_{\nu+1}} \frac{d}{dx}(-x^{-\gamma p})\, dx\right] + l_n^{p(\alpha-\gamma+1)+1}\right\}, \qquad \text{by (8)}$$

$$= O\left\{\int_{l_1}^{l_n} x^{p(\alpha-\gamma+1)}\, dx + l_n^{p(\alpha-\gamma+1)+1}\right\}$$

$$= O\left\{l_n^{p(\alpha-\gamma+1)+1}\right\}$$

in virtue of the conditions imposed on α and γ. So the lemma is proved.

THEOREM 7.[18] *Let (a_n) satisfy (Y_p), $p \geqq 1$. Let $\sum a_n$ be summable (l, r). Let $0 \leqq \kappa < r$ and $s = \sigma + it$. Then $\sum a_n l_n^{-s}$ is summable (l, κ) for*

$$\sigma > \frac{(\alpha+1)(r-\kappa)}{r + \frac{1}{q}}$$

provided $\alpha + 1 \geqq 0$.

Proof. Let $\mu = \dfrac{(\alpha+1)(r-\kappa)}{r+\frac{1}{q}}$. By hypothesis; $A^r(\omega) = o(\omega^r)$. Therefore, by theorem 6, $A^\kappa(\omega) = o(\omega^{\kappa+\mu+\epsilon})$. Since $\alpha + 1 \geqq 0$ we have $\mu + \epsilon > 0$ and so $\sum a_n l_n^{-(\mu+\epsilon)}$ is summable (l, r). Therefore using theorem (F) we see that $\sum a_n l_n^{-(\mu+\epsilon)}$ is summable (l, κ). Since ϵ is arbitrary, the theorem follows.

THEOREM 8. *Let (a_n) satisfy (Y_p) for $p \geqq 1$. Let $\sum a_n l_n^{-\gamma}$ be summable (l, r) where γ is real. Let $s = \sigma + it$ and $0 \leqq \kappa < r$ then $\sum a_n l_n^{-s}$ is summable (l, κ) for*

$$\sigma > \frac{(\alpha+1)(r-\kappa) + \gamma\left(\kappa + \frac{1}{q}\right)}{r + \frac{1}{q}}.$$[19]

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[18] If $0 < r \leqq 1$ we can use theorem 5 and assert summability (l, κ) for $\sigma = \mu$ itself and if $\alpha + 1 > 0$ for $s = \mu + it$ as well (see footnote (6)).

[19] It is to be noted that no restriction is here placed on γ. The only restriction on α is that $\alpha + 1 + \dfrac{1}{p} \geqq 0$ since otherwise all $a_n = 0$.

Proof. Case (i) Let $\alpha - \gamma + 1 < 0$. Let

$$\mu = \frac{(\alpha + 1)(r - \kappa) + \gamma\left(\kappa + \frac{1}{q}\right)}{r + \frac{1}{q}}.$$

Then

$$\mu - \gamma = \frac{(\alpha - \gamma + 1)(r - \kappa)}{r + \frac{1}{q}} > \alpha - \gamma + 1,$$

since $1 > \frac{r - \kappa}{r + \frac{1}{q}} \geqq 0$ and $\alpha - \gamma + 1 < 0$. Therefore $\mu > \alpha + 1$ and consequently by theorem 2, $\sum a_n l_n^{-\mu}$ is absolutely convergent and so summable (l, κ) for $\kappa \geqq 0$. Therefore for $\sigma > \mu$, the series is summable (l, κ) for $\kappa \geqq 0$.

Case (ii). Let $\alpha - \gamma + 1 \geqq 0$. Then $\alpha - \gamma + 1 + \frac{1}{p} > 0$ and by hypothesis $\alpha + 1 + \frac{1}{p} \geqq 0$. Therefore if $b_n = a_n l_n^{-\gamma}$, (b_n) satisfies (Y_p) with $\alpha - \gamma$ in place of α by Lemma 1 and since $\alpha - \gamma + 1 \geqq 0$ we can apply theorem 7 and conclude that

$$\sum b_n l_n^{-s'} \qquad (s' = \sigma' + it)$$

is summable (l, κ) for

$$\sigma' > \frac{(\alpha - \gamma + 1)(r - \kappa)}{r + \frac{1}{q}}.$$

Therefore $\sum a_n l_n^{-s}$ is summable (l, κ) for $\sigma > \sigma' + \gamma$, that is for

$$\sigma > \frac{(\alpha + 1)(r - \kappa) + \gamma\left(\kappa + \frac{1}{q}\right)}{r + \frac{1}{q}}.$$

IV. Part 1

10. Theorem 9.[20] *Let* $f(s) = \sum a_n e^{-l_n s}$ *where the* (a_n) *satisfies* (Y_p) *with* $\alpha = -1$ *that is,*

$$(X_p) \qquad \sum_1^n |a_\nu|^p \left(\frac{l_\nu}{l_\nu - l_{\nu-1}}\right)^p (l_\nu - l_{\nu-1}) = O(l_n)$$

[20] This is the complete theorem mentioned in footnote (2). See (Satz E) of the paper quoted there. The theorem is proved there with the restriction $\frac{l_n}{l_{n-1}} \to 1$. No such restriction is placed on l_n in the above enunciation.

where $p \geqq 1$, so that $f(s)$ converges absolutely for $s > 0$ (s is supposed real and positive here). Let $f(s) \to l$ as $s \to 0$. Then $\sum a_n$ is convergent.

The proof of this theorem is based on the following lemmas.

LEMMA 2. *If (a_n) satisfies (X_p) and $f(s) = O(1)$ as $s \to 0$ then $A(\omega) = O(1)$ as $\omega \to \infty$.*

Proof. If (X_p) holds, so does (X_1) so that we have

$$\sigma_n = \sum_1^n |a_\nu| l_\nu = O(l_n). \tag{9}$$

Let $l_n \leqq \omega < l_{n+1}$. Then,

$$\begin{aligned} |f(s) - A(\omega)| &\leqq \left| \sum_1^n a_\nu (1 - e^{-l_\nu s}) \right| + \left| \sum_{n+1}^\infty a_\nu e^{-l_\nu s} \right| \\ &= S_1 + S_2, \qquad \text{say}. \end{aligned} \tag{10}$$

Now, if $x > 0$, $1 - e^{-x} < x$. So,

$$|S_1| \leqq s \sum_1^n |a_\nu| l_\nu \leqq K s l_n. \tag{11}$$

Also

$$|S_2| \leqq \sum_{n+1}^\infty (\sigma_\nu - \sigma_{\nu-1}) e^{-l_\nu s} = -\sigma_n \frac{e^{-l_\nu s}}{l_\nu} + \sum_{n+1}^\infty \sigma_\nu \Delta\left(\frac{e^{-l_\nu s}}{l_\nu}\right)$$

the transformations employed being valid, since $\sigma_\nu = O(l_\nu)$ and therefore

$$\sigma_\nu \frac{e^{-l_\nu s}}{l_\nu} \to 0 \qquad \text{as} \qquad \nu \to \infty$$

for every $s > 0$.
Therefore,

$$\begin{aligned} |S_2| &\leqq K + K \int_{l_{n+1}}^\infty x \frac{d}{dx}\left(-\frac{e^{-sx}}{x}\right) dx \\ &\leqq K \left\{1 + \int_\omega^\infty e^{-sx} (1 + sx) \frac{dx}{x}\right\} \\ &\leqq K \left\{1 + \int_{\omega s}^\infty e^{-y} (1 + y) \frac{dy}{y}\right\}. \end{aligned} \tag{12}$$

Combining (11) and (12) we get since $l_n \leqq \omega$,

$$|f(s) - A(\omega)| \leqq K \left\{\omega s + 1 + \int_{\omega s}^\infty e^{-y} (1 + y) \frac{dy}{y}\right\}. \tag{13}$$

Taking $\omega s = 1$ so that $\omega \to \infty$ as $s \to 0$ we see that $f(s) - A(\omega) = O(1)$ and therefore $A(\omega) = O(1)$ since $f(s) = O(1)$.

LEMMA 3. *If* $A(\omega) = O(1)$ *and* $f(s) = \sum a_n e^{-l_n s} \to l$ *as* $s \to 0$ *then* $\dfrac{A'(\omega)}{\omega} \to l$ *as* $\omega \to \infty$.

Proof. We may suppose $l = 0$. Since $A(\omega) = O(1)$ we have,

$$f(s) = \sum a_n e^{-l_n s} = s \int_0^\infty A(t)\, e^{-st}\, dt, \text{ for } s > 0.$$

So we have,

$$s \int_0^\infty A(t)\, e^{-st}\, dt \to 0 \text{ as } s \to 0 \text{ and } A(\omega) = O(1).$$

Consequently, using Karamata's method[21] we get,

$$s \int_0^\infty A(t)\, e^{-st}\, g(e^{-st})\, dt \to 0 \text{ as } s \to 0$$

where $g(x)$ is any bounded R-integrable function in $0 \leqq x \leqq 1$. Taking

$$g(x) = 0 \text{ in } 0 \leqq x < e^{-1} \text{ and } g(x) = \frac{1}{x} \text{ in } e^{-1} \leqq x \leqq 1$$

and putting $s = \dfrac{1}{\omega}$ we see that $g(e^{-st}) = e^{st}$ in $0 \leqq t \leqq \omega$ and $g(e^{-st}) = 0$ in $t > \omega$. Therefore

$$\frac{1}{\omega} \int_0^\omega A(t)\, dt \to 0 \text{ as } \omega \to \infty$$

that is,

$$\frac{A'(\omega)}{\omega} \to 0 \text{ as } \omega \to \infty.$$

Proof of theorem 9. Since (a_n) satisfies (X_p) and $f(s) \to l$ as $s \to 0$ we can apply Lemma (2) and conclude that $A(\omega) = O(1)$. Then using Lemma 3, we see that

$$\frac{A'(\omega)}{\omega} \to l \text{ as } \omega \to \infty.$$

Now we can use theorem 4 with $r = 1$, $\alpha = -1$, $S = l$. So we get $A(\omega) = l + o(1)$, that is $A(\omega) \to l$ as $\omega \to \infty$ which is equivalent to the statement that $\sum a_n$ converges.

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[21] J. Karamata "Über die Hardy Littlewoodschen Umkehrung des Abelschen Stetigkeitssatzes," Math. Zeitschrift Band 32 (1930) pp. 319–20.

Part 2

11. THEOREM 10.[22] *Suppose that for every $\delta > 0$, (a_n) satisfies the condition,*

$$Y_p(\delta) \qquad \sum_{\nu=1}^{n} |a_\nu|^p \left(\frac{l_\nu}{l_\nu - l_{\nu-1}}\right)^r (l_\nu - l_{\nu-1}) = O(l_n^{p+1+\delta})$$

where $p \geqq 1$ and as usual

$$\frac{1}{p} + \frac{1}{q} = 1 .$$

Let the function $f(s)$ defined by the series $\sum a_n l_n^{-s}$ which converges absolutely for $\sigma > 1$[23] ($s = \sigma + it$) be regular for $\sigma > \eta$, and suppose that there is a number r such that

$$f(s) = O[|t|^r]$$

uniformly for $\sigma \geqq \eta + \epsilon$, r being positive. Then $\sum a_n l_n^{-s}$ is summable (l, κ) where $0 \leqq \kappa < r$, for

$$\sigma > \frac{r + \frac{\eta}{q}}{r + \frac{1}{q}} - \kappa \cdot \frac{1 - \eta}{r + \frac{1}{q}} .$$

Proof. Let $\epsilon > 0$ be arbitrarily small. Then $f(s)$ satisfies all the conditions of theorem E with $\kappa' = r$, $\kappa = r + \epsilon$. Therefore we conclude that $\sum a_n l_n^{-(\eta+\epsilon)}$ is summable $(l, r + \epsilon)$. Moreover (a_n) satisfies (Y_p) with $\alpha = \frac{\delta}{p}$ so that

$$\alpha + 1 + \frac{1}{p} = 1 + \frac{1 + \delta}{p} > 0$$

(see footnote 19). Therefore we can use theorem 8 with $\gamma = \eta + \epsilon$ and $r + \epsilon$ for r. We thus see that $\sum a_n l_n^{-s}$ is summable (l, κ) if

$$\sigma > \frac{\left(1 + \frac{\delta}{p}\right)(r + \epsilon - \kappa) + (\eta + \epsilon)\left(\kappa + \frac{1}{q}\right)}{r + \epsilon + \frac{1}{q}} .$$

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[22] This corresponds to theorem 9 of (A) which is itself a generalisation of the original Schnee-Landau Theorem.

[23] Condition. $Y_p(\delta)$ is (Y_p) with $\alpha = \frac{\delta}{p}$ and so the series converges absolutely for $\sigma > 1 + \frac{\delta}{p}$ by theorem 2. Since δ is as small as we please, the series converges absolutely for $\sigma > 1$.

But δ, ϵ are as small as we please. So the series is summable (l, κ) if

$$\sigma > \frac{r - \kappa + \eta\left(\kappa + \frac{1}{q}\right)}{r + \frac{1}{q}} = \frac{r + \frac{\eta}{q}}{r + \frac{1}{q}} - \kappa \cdot \frac{1 - \eta}{r + \frac{1}{q}}.$$

In particular the series converge for

$$\sigma > \frac{r + \frac{\eta}{q}}{r + \frac{1}{q}}.$$

THEOREM 11. *Let (a_n) satisfy the condition*[24]

$$Z_p(\delta) \qquad \sum_1^n |a_\nu|^p l_\nu^p (l_\nu - l_{\nu-1}) = O(l_n^{p+1+\delta})$$

for every $\delta > 0$. Suppose further that the sequence (l_n) is subject to the condition

$$(L) \qquad \frac{l_n}{l_n - l_{n-1}} = O(l_n^h) \qquad \text{where } h \geqq 0.$$

Suppose further that the function $f(s)$ defined by the series $\sum a_n l_n^{-s}$ (which will be proved below to converge for $\sigma > h$) satisfies the conditions of theorem 10. *Then $\sum a_n l_n^{-s}$ is summable (l, κ) for*

$$\sigma > \frac{hr + \frac{\eta}{q}}{r + \frac{1}{q}} - \kappa \cdot \frac{h - \eta}{r + \frac{1}{q}} \qquad (0 \leqq \kappa < r).$$

Proof. Let

$$b_n = a_n l_n^{1-h}.$$

Then,

$$\sum_{\nu=1}^n |b_\nu|^p \left(\frac{l_\nu}{l_\nu - l_{\nu-1}}\right)^p (l_\nu - l_{\nu-1}) = O\left\{\sum_1^n |a_\nu|^p l_\nu^{p(1-h)} \cdot l_\nu^{ph} (l_\nu - l_{\nu-1})\right\}$$

$$= O\left\{\sum_{\nu=1}^n |a_\nu|^p l_\nu^p (l_\nu - l_{\nu-1})\right\}$$

$$= O(l_n^{p+1+\delta}) \text{ by } (L) \text{ and } Z_p(\delta).$$

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[24] $Z_p(\delta)$ is the direct generalisation of the condition imposed on (a_n) in the original Schnee-Landau Theorem. The sequence (l_n) also satisfies (L) in the original theorem. If $a_n = O(l_n^\delta)$ $Z_p(\delta)$ is satisfied for all $p \geqq 1$ and so as in foot note (16) the original Schnee-Landau theorem may be obtained as a particular case of Theorem 11 by putting $q = 1$, $p = \infty$.

Therefore (b_n) satisfies the condition of theorem 10. Consequently

$$\sum b_n l_n^{-s'} = F(s'), \qquad s' = \sigma' + it$$

is absolutely convergent for $\sigma' > 1$ and therefore $\sum a_n l_n^{-s}$ is absolutely convergent for $\sigma > 1 + h - 1 = h$. Moreover $F(s')$ is regular for $\sigma' > \eta + 1 - h$ since $f(s)$ is so, for $\sigma > \eta$ and $F(s') = O[|t|^r]$, since $f(s)$ satisfies this relation. So by theorem 10, $\sum b_n l_n^{-s'}$ is summable (l, κ), $0 \leqq \kappa < r$ for

$$\sigma' > \frac{r + \dfrac{\eta + 1 - h}{q}}{r + \dfrac{1}{q}} - \kappa \cdot \frac{1 - (\eta + 1 - h)}{r + \dfrac{1}{q}}.$$

Hence $\sum a_n l_n^{-s}$ is summable (l, κ) for

$$\sigma > \frac{hr + \dfrac{\eta}{q}}{r + \dfrac{1}{q}} - \kappa \cdot \frac{h - \eta}{r + \dfrac{1}{q}}.$$

12. *Example.* First, let

$$\psi(s) = 1^{-s} - 2^{-s} + 3^{-s} - \cdots = (1 - 2^{1-s})\,\zeta(s)$$

where $\zeta(s)$ is the Zeta function of Riemann. Then $\psi(s)$ has the same μ-function as $\zeta(s)$ so that if the latter be denoted by $\mu(\sigma)$ we have $\mu(\sigma) = \frac{1}{2} - \sigma$ for $\sigma < 0$ so that,

$$\frac{\mu(\sigma)}{\sigma} \to -1 \text{ as } \sigma \to -\infty. \tag{14}$$

Now construct a Dirichlet's series as follows.

$$\begin{cases} \text{(i)} \quad f(s) = \Sigma\, a_n l_n^{-s}\,; \qquad \text{(ii)} \quad l_n = n \\ \text{(iii)} \quad a_n = (-1)^{n-1}, \text{ if } n \neq \text{a perfect cube}\,; \\ \text{(iv)} \quad a_n = (-1)^{n-1}\, n^{\frac{2}{3p}} \text{ if } n = \text{a perfect cube}, \end{cases} \tag{15}$$

where $p > 1$. Then $Y_p(\delta)$ becomes,

$$Y_p(\delta) \qquad \sum_{\nu=1}^{n} |\, a_\nu \,|^p\, \nu^p = O(n^{p+1}) = O(n^{p+1+\delta}).$$

Now

$$f(s) = \Sigma\, a_n n^{-s} = \psi(s) - \psi(3s) + \psi\left(3s - \frac{2}{p}\right). \tag{16}$$

Let $\nu(\sigma)$ be the μ-function of $f(s)$. Then since it is positive decreasing function of σ we have in virtue of (16)

$$\nu(\sigma) = \mu\left(3\sigma - \frac{2}{p}\right). \tag{17}$$

So,

$$\frac{\nu(\sigma)}{\sigma} \to -3 \text{ as } \sigma \to -\infty . \tag{18}$$

If $s = \sigma + it$ then, by the definition of the μ-function:

$$f(s) = O[\,|\,t\,|^{\nu(\sigma)+\epsilon}]$$

and so by applying theorem 10 we get, if $\sigma(r)$ denotes the abscissae of summability $(l, r) \equiv (n, r)$,

$$\sigma(\kappa) \leqq \frac{\nu(\sigma) + \frac{\sigma}{q}}{\nu(\sigma) + \frac{1}{q}} - \kappa \cdot \frac{1-\sigma}{\nu(\sigma) + \frac{1}{q}} \leqq \frac{-3 + \frac{1}{q}}{-3} - \kappa \cdot \frac{-1}{3} = 1 - \frac{1}{3q} - \frac{\kappa}{3}. \tag{19}$$

Now, $a_n = O\left(n^{\frac{2}{3p}}\right)$ and using the analogue of theorem 8, with $q = 1$, $(p = \infty)$, we get

$$\sigma(\kappa) \leqq \frac{\left(1 + \frac{2}{3p}\right)(\nu(\sigma) - \kappa) + \sigma(\kappa + 1)}{\nu(\sigma) + 1} \leqq \frac{2}{3} + \frac{2}{3p} - \frac{\kappa}{3}. \tag{20}$$

From (19) and (20) we get,

$$1 - \frac{1}{3q} - \frac{\kappa}{3} - \left(\frac{2}{3} + \frac{2}{3p} - \frac{\kappa}{3}\right) = -\frac{1}{3p} < 0 . \tag{21}$$

(21) shows that the abscissae of summability as obtained by using the result in this paper is less than that obtained by the use of analogous results in (A) and therefore theorem 10 gives better information than when (a_n) is subjected to conditions of the type in theorem (A).

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THE RANGES OF ANALYTIC FUNCTIONS

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Let $f(z)$ be a function analytic for $|z| < 1$. The set of values assumed by $f(z)$ in $|z| < 1$, i.e. its range, has been the subject of numerous investigations. These investigations have been concerned with families of functions, which, because of certain restrictions, (such as $f(0) = 0$, $|f'(0)| \geqq 1$), have no limiting functions, (in the sense of uniform convergence in every closed subregion of $|z| < 1$), which are identically constant. Montel[2] investigated the general case of families of non-constant functions having no constant limiting functions, obtaining results which contain results obtained by Bloch, Bohr, Fekete, Koebe, Landau, Valiron. In this paper, certain more inclusive families of functions will be considered which may have constant limiting functions.

1. Families of meromorphic functions

We shall need some results which generalize and apply those obtained in this subject in a previous paper.[3] In the following, if $\{f_n(z)\}$ is a sequence of functions meromorphic for $|z| < 1$, and if with each function is associated an open arc on $|z| = 1$, we denote by $s\{f_n\}$, $S\{f_n\}$ respectively the cluster set[4] of the sequence with respect to the associated arcs and the cluster set of the sequence on the arcs. The range of a function $f(z)$ meromorphic for $|z| < 1$ is defined as the open connected point set in the $w = f(z)$-plane assumed by $f(z)$ in $|z| < 1$. The range is said to cover a point set E at least n times if every point in E is assumed at least n times by $f(z)$ in $|z| < 1$.

THEOREM 1. *Let $\{f_n(z)\}$ be a sequence of functions meromorphic for $|z| < 1$ and let each function have associated with it an arc on $|z| = 1$. Suppose that there is a point α contained in $s\{f_n\}$ but not in $S\{f_n\}$. Let D be the domain containing α and bounded only by points of $S\{f_n\}$.*

(a) *If α is omitted by the sequence, at most one other point of D is omitted. If α and any other point of D are omitted, no third point of the extended plane is omitted.*

(b) *If α and a second point β[5] are omitted by the sequence, and if $f_n(z)$ assumes a point w $n(w)$ times in $|z| < 1$,*

$$\limsup_{n \to \infty} n(w) = \infty$$

if $w \neq \alpha, \beta$, is in D.

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[1] National Research Fellow.

[2] Annales de l'Ecole Normale Supérieure, Series 3, vol. 46, 1929, pp. 1–23.

[3] Transactions of the American Mathematical Society, vol. 35, (1933), pp. 433–442. This paper will be referred to as "T".

[4] Cf. T, §2, for these definitions.

[5] The point β may or may not be in D.

(a) Part (a) is simply Lemma 4.2 of T.

(b) If (b) is not true, there is a point $\gamma \neq \alpha, \beta$, in D, and an integer N such that $f_n(z)$ assumes the value γ at most N times in $| z | < 1$. We can suppose that $\alpha = 0, \beta = \infty, \gamma = 1$, transforming every function of the sequence by a linear transformation to bring this about if it is not true already. By hypothesis $f_n(z) \neq 0, \infty$, for large $n, n > p$. Consider the sequence $\{\varphi_n(z)\}$,

$$\varphi_n(z) = [f_{n+p}(z)]^{1/N+1}, \qquad (n = 1, 2, \cdots)$$

choosing one of the $N + 1$ single-valued branches, as described below. The sequence $\{\varphi_n(z)\}$ omits the values $0, \infty$. Moreover if $\varphi_n(z)$ assumed every $(N + 1)^{st}$ root of unity, $f_{n+p}(z)$ would assume $\gamma = 1$ $N + 1$ times. Then $\varphi_n(z)$ can be chosen as a branch of $f_{n+p}(z)$ which does not assume $\gamma' = 1$. Evidently $\alpha' = 0$ is a point of $s\{\varphi_n\}$, (where we associate the same arc with $\varphi_n(z)$ as with $f_{n+p}(z)$), but not of $S\{\varphi_n\}$. Let D' be the domain containing α' and bounded only by points of $S\{\varphi_n\}$. By (a), D' cannot contain the point $\gamma' = 1$ which the sequence $\{\varphi_n(z)\}$ omits. This statement is self-contradictory. For there is a continuous curve A in D which joins $\alpha = 0$ to $\gamma = 1$. Let w_0 be a point on A so close to $\alpha = 0$ that the $N + 1$ values of $w_0^{1/N+1}$ are all in D' (which contains a neighborhood of $\alpha' = 0$). Then find the image A' of that part of A between w_0 and $\gamma = 1$ in the w'-plane, $w' = w^{1/N+1}$, by continuing analytically a branch of $w^{1/N+1}$ along A from w_0 to $\gamma = 1$. We can choose that branch giving $\gamma' = 1$ as the image of $\gamma = 1$. Then since D' does not contain $\gamma' = 1$, A' must have a point w' in common with $S\{\varphi_n\}$. But then A has the point $w = w'^{N+1}$ in common with $S\{f_n\}$, contradicting the fact that A lies in D.

THEOREM 2. *Let $\{f(z)\}$ be a family of functions meromorphic for $| z | < 1$ and let each function have associated with it an arc on $| z | = 1$. Denote by s the product of all the sets $s\{f_n(z)\}$ for all sequences of functions from the family and let S be the sum of all the sets $S\{f_n\}$.*[6] *Then S is closed. Suppose that there is a point α belonging to s but not to S, and let D be the domain containing α and bounded only by points of S. Then if D_1, D_2 are any two mutually exclusive closed point sets in the extended plane, neither of which contains α, and such that $D_1 \subset D$, and if q is an arbitrary positive integer, there is a neighborhood D_0 of α, depending only on D_1, D_2, q, such that if $f(z)$ is in the family, its range covers D_1 at least q times if it does not cover either D_0 or D_2.*

If we choose D_1, D_2 each to contain open regions, we see that a number k exists with the property that if $f(z)$ is in the family, its range includes the interior of some circle of radius k.

That S is closed was shown in the note to the definition. If the theorem is not true, there exist sets D_1, D_2 with the properties described, a positive integer

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[6] The sequence $\{f_n(z)\}$ is not supposed necessarily to consist of distinct functions. The set S is then the set of all cluster values of functions of the family at points of their associated arcs, closed by the addition of its limit points.

q, three sequences of points $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and a sequence $\{f_n(z)\}$ of functions of the family such that

$$f_n(z) \neq \alpha_n, \gamma_n, \qquad (n = 1, 2, \cdots)$$

and that $f_n(z) = \beta_n$ at most $q - 1$ times in $| z | < 1$, where $\alpha_n \to \alpha$ and where β_n, γ_n are in D_1, D_2 respectively. We can suppose that proper subsequences have already been chosen, if necessary, so that the sequences $\{\beta_n\}$, $\{\gamma_n\}$ converge, to β, γ in D_1, D_2 respectively. We can also suppose that $\alpha = 0, \beta = 1, \gamma = \infty$, transforming every function of the family by the same linear transformation to bring this about if it is not true already. The sequence $\{\varphi_n(z)\}$, where

$$\varphi_n(z) = \frac{\beta_n - \gamma_n}{\beta_n - \alpha_n} \cdot \frac{f_n(z) - \alpha_n}{f_n(z) - \gamma_n}$$

is a sequence of functions omitting the values $\alpha = 0, \gamma = \infty$, and $\varphi_n(z)$ assumes the value $\beta = 1$ at most $q - 1$ times. Associate the same arc on $| z | = 1$ with $\varphi_n(z)$ as with $f_n(z)$. Evidently $s\{f_n\} = s\{\varphi_n\}$, $S\{\varphi_n\} = S\{f_n\} \subset S$. Then if D' is the domain containing $\alpha = 0$ and bounded only by points of $S\{\varphi_n\}$, $D' \supset D$, and so $D_1 \subset D'$. The sequence $\{\varphi_n(z)\}$ omits α and γ. Moreover $\alpha \in s\{\varphi_n\}$, so that by Theorem 1 (b), there is an infinite subsequence of $\{\varphi_n(z)\}$ each function of which takes on $\beta = 1$ at least q times, contrary to the assumptions made in the proof.

2. Functions with the property $K(\rho)$

In the following, we specialize these results. Let $f(z)$ be analytic for $| z | < 1$, and let A be an open arc on $| z | = 1$. Suppose that $f(0) = 0$ and that if $\{P_n\}$ is a sequence of points converging to a point of A,

$$\liminf_{n \to \infty} | f(P_n) | \geqq 1 .$$

Then if $mA \geqq \rho > 0$, $f(z)$ will be said to have the property $K(\rho)$. The family of functions with the property $K(\rho)$ has the identically constant limit function 0. For z^n has the property $K(\rho)$ for every value of $\rho \leqq 2\pi$ and if $| z | < 1$, $\lim_{n \to \infty} z^n = 0$.

THEOREM 3. *The point $w = 0$ is the only point common to all the ranges of functions having the property $K(\rho)$, if ρ is fixed, $\rho < 2\pi$.*

If w_0 belongs to the range of every function with the property $K(\rho)$, so does the circle $| w | \leqq | w_0 |$. For if $f(z)$ has the property $K(\rho)$, the function $\lambda f(z)$ also has this property, if $| \lambda | \geqq 1$, so the range of $\lambda f(z)$ also contains w_0. Then the range of $f(z)$ contains w_0/λ as was to be proved. Thus either the theorem is true or there is a neighborhood of the origin covered by the range of every function with the property $K(\rho)$. But consider the function

$$f_n(z) = \frac{\varphi_n(z) - e^{-n}}{e^{-n}\varphi_n(z) - 1}, \qquad \varphi_n(z) = e^{n\frac{z+1}{z-1}} .$$

The function $\varphi_n(z)$ has modulus 1 on $| z | = 1$ except at $z = 1$, and when $| \varphi_n(z) | = 1$, $| f_n(z) | = 1$, so that $| f_n(z) | = 1$ on $| z | = 1$ except at $z = 1$.

Since $f_n(0) = 0$, $f_n(z)$ has the property $K(\rho)$, if $\rho < 2\pi$. But $f_n(z) \neq e^{-n}$ so that to an arbitrary neighborhood of $w = 0$ corresponds a value of n such that the range of $f_n(z)$ does not cover the neighborhood, which proves the theorem.

THEOREM 4. (a) *If D_1 is an arbitrary closed point set in $| w | < 1$, not containing $w = 0$ and q an arbitrary integer, there is a neighborhood D_0 of $w = 0$, depending only on D_1, ρ, q such that if $f(z)$ has the property $K(\rho)$ $\rho > 0$, its range covers D_1 at least q times if it does not cover D_0.*

(b) *There is a positive function $k(\rho)$ depending only on $\rho > 0$ such that if $f(z)$ has the property $K(\rho)$, its range covers the interior of some circle of radius $k(\rho)$.*

(a) The family of functions with the property $K(\rho)$ is a family to which Theorem 2 can be applied. In the terminology of that theorem, $w = 0$ is a point of s, and the set S is outside $| w | < 1$. Part (a) then follows at once if the set D_2 is taken as the point ∞.

(b) Part (b) is obvious from (a) if we take D_1 as a set containing an open region. The determination of a function with the properties of $k(\rho)$ as described in (b) has some interest. In T, the results which lead to Theorem 1 of this paper can be made numerical. This then yields an evaluation of $k(\rho)$. An evaluation of $k(\rho)$ could be obtained at once if the theorems of this paper could be deduced from the theorem of Bloch.[7] Such a deduction would be impossible if to every integer n there corresponds a function $f_n(z)$ with the property $K(\rho)$ and satisfying the inequality

$$(1 - | z |) | f'(z) | \leqq 1/n .$$

It is plausible that such functions exist for all n. In any case a proof by Bloch's theorem would not yield the precise information given in Theorem 4 and below. It will be seen in §3 how the theorem of Bloch can be deduced from Theorem 4.

Theorem 4 can be made more precise as follows.

THEOREM 4′. *There are positive functions $\eta_1(\rho, q, r)$, $\eta_2(\rho, q, r)$ depending only on the stated arguments, where η_1, $-\eta_2$ are monotone non-decreasing in r for ρ, q fixed,*

$$\lim_{r \to 0} \eta_1(\rho, q, r) = 0 , \qquad \eta_2(\rho, q, 0) > 0 , \qquad \eta_2(\rho, q, r) < 1 - r ,$$

and such that if $f(z)$ has the property $K(\rho)$ and assumes the value α, $0 \leqq | \alpha | < 1$, at most $q \geqq 0$ times, the range of $f(z)$ covers the circle $| w | \leqq r = | \alpha |$ except possibly for points in $| w - \alpha | < | \alpha | \eta_1$ and also covers the circle $| w - \alpha | < \eta_2$.

The first part of the theorem is equivalent to the statement that to an arbitrary positive number ϵ corresponds a number $R_\rho(\epsilon)$ depending only on ρ, ϵ, such that if $f(z)$ has the property $K(\rho)$, its range covers $| w | < R_\rho(\epsilon)$ except possibly for points in a circle subtending an angle of magnitude less than ϵ at $w = 0$. The proof depends on the following lemma.

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[7] This theorem states that there is an absolute constant $B > 1/6$ with the property that if $f(z)$ is analytic for $| z | < 1$, and if $| f'(0) | \geqq 1$, the range of $f(z)$ contains the interior of a circle of radius B. There is even a positive number $B' < B$ such that $f(z)$ maps some domain in $| z | < 1$ in a one to one way on the interior of a circle of radius B'. It would be interesting to prove the corresponding theorem for functions with the property $K(\rho)$.

LEMMA. *There is no sequence of functions with the property $K(\rho)$, for a fixed value of $\rho > 0$, and sequences $\{\alpha_n\}$, $\{\beta_n\}$, of points in $|w| < 1$, $\alpha_n \to 0$, $\beta_n \to 0$, where $f_n(z)$ omits β_n and assumes α_n at most q times (q independent of n) and where*

$$\liminf_{n\to\infty} \left| 1 - \frac{\alpha_n}{\beta_n} \right| > 0 .$$

Suppose the lemma were not true. Consider the sequence $\{\varphi_n(z)\}$,

$$\varphi_n(z) = \frac{f_n(z) - \beta_n}{\alpha_n - \beta_n} .$$

The function $\varphi_n(z)$ omits the value 0 and assumes the value 1 at most q times. The sequence $\{\varphi_n(z)\}$ is therefore normal.[8] We shall suppose that the sequence $\{f_n(z)\}$ has already been so selected that the sequence $\{\varphi_n(z)\}$ converges uniformly in every closed subregion of $|z| < 1$ to a function $\varphi(z)$, and so that if the arc A_n on $|z| = 1$ is associated with the function $f_n(z)$, the sequence of arcs approaches a limiting arc A. By rotating the z-plane suitably (the rotation depending on n) and by shrinking the arcs A_n if necessary, we can even suppose that the arc A_n coincides with the arc A, for all values of n.

Since $\varphi_n(0) = \left(1 - \frac{\alpha_n}{\beta_n}\right)^{-1}$ it follows that $\limsup_{n\to\infty} |\varphi_n(0)|$ is finite. The function $\varphi(z)$ is then not the constant ∞. Let $\{z_n\}$ be a sequence of points in $|z| < 1$ approaching a point P of A, and suppose that $\lim_{n\to\infty} \varphi(z_n) = w_0$.

(i) If $w_0 = \infty$ for every such sequence $\{z_n\}$, the function $1/\varphi(z)$ can be continued analytically across A, by the Schwarz reflection principle. The resulting function vanishes identically on A and so throughout $|z| < 1$, contradicting the fact that $\varphi(z)$ is analytic in $|z| < 1$.

(ii) There is then a sequence $\{z_n\}$ such that w_0 is finite. Associate the arc $A_n = A$ with the function $\varphi_n(z)$. Evidently $S\{\varphi_n\}$ is the point ∞. We shall show that the set $s\{\varphi_n\}$ includes the points 0 and 1 by showing that it includes the whole finite plane. To every value of n an integer a_n can be chosen so that

$$|\varphi_{a_n}(z_n) - \varphi(z_n)| \leqq 1/n .$$

Then $\lim_{n\to\infty} \varphi_{a_n}(z_n) = w_0$, and w_0 belongs to the set $s_1\{\varphi_n\}$, the strong cluster set of the sequence $\{\varphi_n(z)\}$ with respect to the arcs $\{A_n\}$.[9] Then the entire finite plane belongs to $s_1\{\varphi_n\} \subset s\{\varphi_n\}$.[10] That the function $\varphi_n(z)$ should omit the values 0, ∞ and assume the value 1 at most q times is impossible in accordance with Theorem 1. The lemma is thus completely proved.

We can now prove Theorem 4′. The first part of the theorem, that involving $\eta_1(\rho, q, r)$, is an immediate consequence of the lemma, except for the monotone

[8] P. Montel, Leçons sur les familles normales, Paris, 1927, pp. 69–70.

[9] Defined in T, §2.

[10] T, Lemma 4.4.

character of $\eta_1(\rho, q, r)$ in r. If we choose the best possible function for η_1 (i.e. the smallest), it is seen at once, by considering $\lambda f(z)$, for λ real, $\lambda > 1$, that if $\lambda r < 1$,

$$\eta_1(\rho, q, \lambda r) \geqq \eta_1(\rho, q, r) .$$

If we take $\alpha_n = 0$ for all n in the lemma, we have the second part of the theorem (that involving η_2), for $\alpha = 0$. The theorem is then proved completely by transforming $f(z)$ linearly to reduce the general case to this one.

A function with the property $K(\rho)$ will be said to have the property $K_\sigma^{(q)}(\rho)$, $0 < \sigma < \pi$, $0 \leqq q$, if the set of those points in $| w | < 1$ assumed at least q times by $f(z)$ has measure not greater than σ.

THEOREM 5. (a) *If $\sigma < \pi$, there is a point set $E_\sigma^{(q)}(\rho)$ depending only on ρ, σ, q and containing some neighborhood of the origin such if $f(z)$ has the property $K_\sigma^{(q)}(\rho)$, its range covers $E_\sigma^{(q)}(\rho)$.*

(b) *The family of functions with the property $K_\sigma^{(1)}(\rho)$ is compact in the sense that it is a normal family which contains all its limit functions, if $\sigma < \pi$.*

(c) *If σ_0 is the lower limit of the values of σ for which the family considered in* (b) *is not empty, $\sigma_0 > 0$, and there is a function with the property $K_{\sigma_0}^{(1)}(\rho)$.*

(a) Part (a) shows that there was not so much choice as might have been expected in the sequence used above of functions with the property $K(\rho)$ which omit points approaching the origin. If $f(z)$ has the property $K_\sigma^{(q)}(\rho)$, it assumes the values in a point set E of measure not less than $\pi - \sigma$ at most $q - 1$ times. Choose $\epsilon > 0$ so small that $\pi\epsilon^2 \leqq (\pi - \sigma)/3$, $\pi(2\epsilon - \epsilon^2) \leqq (\pi - \sigma)/3$. Then if D_1 is the ring $\epsilon \leqq | w | \leqq 1 - \epsilon$, D_1 must contain a subset of E of measure at least $(\pi - \sigma)/3$. Then (a) follows at once from Theorem 4 (a). It would be interesting to describe $E_\sigma^{(q)}(\rho)$ more fully. It is easy to see that it consists of all the points on a certain set of circles concentric with the origin.

(b) If the family of functions with the property $K_\sigma^{(1)}(\rho)$ were not normal, it would contain a sequence $\{f_n(z)\}$ no subsequence of which converged uniformly in every closed subregion of $| z | < 1$. No subsequence of $\{f_n(z)\}$ can omit two points of the finite plane,[11] so the sequence contains a subsequence which assumes every point in the finite plane, with one possible exception.[12] We can suppose that this is already true of the sequence $\{f_n(z)\}$. Then if $R(f_n)$ is the range of $f_n(z)$, and if D is the circle $| w | < 1$,

$$m\,[R(f_n \cdot D] \geqq m\left[\prod_{j=n}^{\infty} R(f_j) \cdot D\right].$$

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[11] P. Montel, Leçons sur les familles normales, Paris, 1927, pp. 61–64.

[12] We recall that a sequence is said to assume a value if at most a finite number of functions of the sequence do not assume that value. In the case considered, if there is no subsequence of $\{f_n(z)\}$ which omits a value of the finite plane, the desired subsequence can be taken as the sequence $\{f_n(z)\}$ itself; if there is such a subsequence (not taking on some finite value), it will serve as the desired subsequence.

The set on the right increases toward D less possibly one point, as n becomes infinite, so

$$\lim_{n\to\infty} m[R(f_n)\cdot D] = mD = \pi$$

contradicting the hypothesis that $m[R(f_n)\cdot D] \leqq \sigma < \pi$. The family considered is thus normal. Let $\{f_n(z)\}$ be a sequence of functions with the property $K_\sigma^{(1)}(\rho)$ converging uniformly in every closed subregion of $|z| < 1$ to the function $f(z)$. We can assume that the length of the arc A_n associated with $f_n(z)$ is ρ (taking a subarc if necessary). Moreover, choosing a subsequence of $\{f_n(z)\}$ if necessary, we can suppose that the arcs $\{A_n\}$ approach a limiting arc A, and even that the arc A_n coincides with the arc A for all values of n, rotating the z-plane suitably. Assume that the original sequence $\{f_n(z)\}$ and its corresponding arcs have these properties already. We show that $f(z)$ has the property $K(\rho)$. Since $f_n(0) = 0$, $f(0) = 0$. Let $\{P_n\}$ be a sequence of points approaching a point of A. It is possible to find an integer a_n so large that

$$|f_{a_n}(P_n) - f(P_n)| \leqq 1/n\,.$$

Then the limiting values of the sequences $\{f_{a_n}(P_n)\}$, $\{f(P_n)\}$ are the same, and are strong cluster values of the sequence $\{f_n(z)\}$ with respect to the arcs $\{A_n\}$.[13] If any one of these values is in $|w| < 1$, every point of $|w| < 1$ is a cluster value of the sequence with respect to these arcs, since $S(f_n)$ is outside $|w| < 1$.[14] By an argument similar to that already used in this proof, this implies that

$$\limsup_{n\to\infty} m[R(f_n)\cdot D] = \pi$$

contradicting the hypothesis that $m[R(f_n)\cdot D] \leqq \sigma < \pi$. Then $\liminf_{n\to\infty} |f(P_n)| \geqq 1$ and $f(z)$ has the property $K(\rho)$. Moreover by a classical theorem of Hurwitz, if a point belongs to $R(f)$, it belongs also to $R(f_n)$ for all large values of n, so that

$$m[R(f)\cdot D] \leqq \lim_{n\to\infty} m\left[\prod_{j=n}^{\infty} R(f_j)\cdot D\right] \leqq \sigma_0.$$

Then $f(z)$ has the property $K_\sigma^{(1)}(\rho)$, and (b) is proved.

(c) There is a sequence of functions $\{f_n(z)\}$ with the property $K(\rho)$ such that

$$\lim_{n\to\infty} m[R(f_n)\cdot D] = \sigma_0.$$

Since the limiting functions of this sequence (existing by (b)) all have the property $K_\sigma^{(1)}(\rho)$ for some value of σ, it follows from (b) that σ can be taken as σ_0. The number σ_0 depends only on ρ. If $f(z)$ has the property $K(\rho)$, $mR(f) \geqq \sigma_0$.

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[13] Defined in T, §2.

[14] T, Lemma 4.4.

3. Generalizations

There are certain desirable generalizations which will be found to include the results of Bloch and Montel. Let $f(z)$ be analytic for $|z| < 1$. If there is a function $\varphi(\zeta)$ analytic for $|\zeta| < 1$, and mapping $|\zeta| < 1$ in a one to one way on a domain in $|z| < 1$, and if $f[\varphi(\zeta)]$ has the property $K(\rho)$ as defined in §2, $f(z)$ will still be said to have the property $K(\rho)$. The range of $f[\varphi(\zeta)]$ is of course a subset of the range of $f(z)$, so that the theorems of §2 are still true, except Theorem 5 (b) and (c).

THEOREM 6. *Let $f(z)$ be a function analytic for $|z| < 1, f(0) = 0$.* (a) *Let A be a Jordan arc in $|z| \leqq 1$ with the property that when $\{P_n\}$ is any sequence of points approaching a point of A, $\liminf_{n\to\infty} |f(P_n)| \geqq 1$. If θ is the angle formed by all the radii of $|z| = 1$ which intersect A, $f(z)$ has the property $K(\rho)$, where*

$$\rho = 2^{-(2\pi/\theta+1)}.$$

(b) *If*

$$\max_{|z|=r} |f(z)| \geqq 1,$$

$f(z)$ has the property $K(\rho)$, if

$$\rho < 4 \arcsin \frac{1}{\sqrt{2}} \frac{1-r}{1+r}.$$

(a) Let ν be the smallest positive integer for which $\pi/\nu \leqq \theta$. Then if $\theta < \pi$, $\nu > 1, \pi/(\nu-1) > \theta$

$$\nu < \frac{\pi}{\theta} + 1,$$

and if $\theta \geqq \pi, \nu = 1$, so the inequality still holds. It is readily seen that the arc A contains a subarc $\mathfrak{A}$ with the following two properties. (i) If P, Q are the endpoints of $\mathfrak{A}$, $\mathfrak{A}$ does not intersect the diameters through P, Q, except at P, Q. (ii) Let O be the origin. Then every radius in the angle POQ, which is of magnitude π/ν, meets A. Let the radii through P, Q meet $|z| = 1$ in S, R respectively. Let $z = \varphi(\zeta)$ map $|\zeta| < 1$ in a one to one and conformal manner on $|z| < 1$ less the curvilinear figure $PQRS$. By the theorem of Osgood and Carathéodory, $\varphi(\zeta)$ takes on continuous boundary values on $|\zeta| = 1$.[15] The rectangle $PQRS$ does not contain O so we can normalize the map by setting $\varphi(0) = 0, \varphi(1) = P$. The arc $\mathfrak{A}$ corresponds to an arc of length ρ on $|\zeta| = 1$ with one endpoint at $\zeta = 1$. Then the function $f[\varphi(\zeta)]$ has the property $K(\rho)$. We prove (a) by establishing the inequality

$$\rho \geqq 2^{-(2\nu-1)} \geqq 2^{-(2\pi/\theta+1)}.$$

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[15] Cf. for instance L. Bieberbach, Lehrbuch der Funktionentheorie, vol. 2, 2d edition, 1931, pp. 29–31.

Let $\psi(z)$ be the inverse of $\varphi(\zeta)$. Then $\psi(0) = 0$. By a familiar method of Lindelöf,[16] since $\psi(z) - 1$ is bounded in $|z| < 1$, where defined:

$$|\psi(z) - 1| < 2,$$

and since on $\mathfrak{A}$,

$$|\psi(z) - 1| < \rho,$$

it follows that

$$1 = |\psi(0) - 1|^{2^\nu} \leqq 2^{2^\nu - 1}\rho.$$

Then $\rho \geqq 2^{-(2^\nu - 1)} \geqq 2^{-(2\pi/\theta + 1)}$ as was to be proved.

(b) By hypothesis there is a point z_0, $|z_0| = r$, at which $|f(z)| = \lambda \geqq 1$. There is a curve through z_0 on which $|f(z)| = \lambda$. Trace this curve A from z_0, choosing any branch at a point where there are several branches (i.e. at a zero of $f'(z)$).

(i) If A intersects itself there is a domain D in $|z| < 1$ bounded only by points of A. Then $|f(z)| \leqq \lambda$ in D. If $f(z)$ did not vanish in D we should also have $|1/f(z)| \leqq 1/\lambda$ in D, i.e. $|f(z)| \equiv \lambda$ in D. Then $f(z)$ would be a constant of modulus λ, contradicting the fact that $f(0) = 0$. There is thus a zero of $f(z)$ in D, and $f(z)$ then evidently has the property $K(\rho)$ for any $\rho \leqq 2\pi$.

(ii) If A does not intersect itself, it is seen at once that it has limit points on $|z| = 1$ (and in fact if z traces A from z_0 that $|z| \to 1$). Then if $|z_0| = r < \rho < 1$, consider the function $\varphi_\rho(z) = f(\rho z)$, analytic for $|z| < 1$ and vanishing for $z = 0$. On a simple Jordan arc J leading from a point of $|z| = r/\rho$ to a point of $|z| = 1$, $|\varphi_\rho(z)| = \lambda$. Now by a theorem of Fenchel,[17] if $|\zeta| < 1$ is mapped in a one to one and conformal way on $|z| < 1$ less J, so that $\zeta = 0$ goes into $z = 0$, the arc on $|\zeta| = 1$ corresponding to J has measure not less than

$$\xi(r, \rho) = 4 \arcsin \frac{1}{\sqrt{2}} \frac{1 - r/\rho}{1 + r/\rho}.$$

Then $f(z)$ has the property $K(\xi(r, \rho))$. Letting ρ approach 1, (b) is proved.

The relation between this work and that of Bloch and Montel is fairly close, although the methods are different. Bloch considered families of functions $f(z)$ analytic for $|z| < 1$, $f(0) = 0$, $|f'(0)| \geqq 1$. Let $M_r(f)$ be the maximum of $|f(z)|$ on $|z| = r$. Then

$$1 \leqq \frac{1}{2\pi} \int_{|z|=r} \frac{|f(z)|}{|z|^2} |dz| \leqq \frac{M_r(f)}{r},$$

so that $M_r(f)$ is bounded below independently of $f(z)$. Montel considered families of non-constant functions analytic for $|z| < 1$ which have no limiting func-

[16] A similar discussion is carried out in L. Bieberbach, Lehrbuch der Funktionentheorie, vol. 2, 2d edition (1931), pp. 20–21.

[17] Göttinger Nachrichten, 1930, p. 125.

tions which are indentically constant. A necessary condition for this is that if $f(z)$ is in the family,

$$M_r[f(z) - f(0)]$$

be bounded below independently of $f(z)$. The theorems of this paper have been seen to apply to such families (by Theorem 6 (b)). The results are less strong than those obtained by Montel in that the discussion of the ranges is limited to that part of the ranges in $| w | < m$, where m is the greatest lower bound of $M_r[f(z) - f(0)]$ for some fixed r for all the functions of the family, and it can be seen from simple examples that the results of Montel are not true for the more general families of functions discussed in this paper.

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A MATHEMATICAL LOGIC WITHOUT VARIABLES. I

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In proving intuitive theorems about formal logics it sometimes becomes necessary to analyze the structures and processes which can occur in the logic. It is therefore convenient to perform this analysis once and for all in as general a way as possible. Also the analysis should be as complete as possible.

In this paper we carry still further Curry's[1] analysis in that we not only avoid the use of variable atomic structures but also the use of variable atomic processes; e.g., processes like "If **If**, then **fx** if **f** and **x** are well-formed" in which the result of the process is a matter of choice.

This naturally necessitates a number of changes in Curry's system. In order that the fundamental theorem (MV of this paper or Theorem 5 of Curry 1932[2]) remain true without $BI = I$ being a provable theorem still more changes are necessary. We also get along without the K-function by making some omissions; a rather unimportant change.

As the paper is written it gives an analysis of the formal logic which results if we take Church's five rules of procedure[3] and the postulate '$x \cdot x = x$. In Section H we illustrate how the results of the paper may be used to provide an analysis of Church's first three rules of procedure; a result which it is anticipated will be useful.

Section A

We shall use the following undefined terms:

$$(\ ),\ \Pi,\ \Sigma,\ I,\ J\ .$$

) is called a "right parenthesis" and (is called a "left parenthesis." (**FX**) is to have exactly the meaning that Church assigns to {**F**}(**X**), and hence agrees with Curry's usage.

Π and Σ have the same significance as in Church 1932 (see pp. 351–353).

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[1] H. B. Curry, *Grundlagen der kombinatorischen Logik*, Am. Jour. Math., Vol. 52 (1930), pp. 509–536, and pp. 789–834. We shall refer to this paper as "Curry 1930."

[2] H. B. Curry, *Some Additions to the Theory of Combinators*, Am. Jour. Math., Vol. 54 (1932), pp. 551–558. Referred to as "Curry 1932."

[3] In A. Church, *A set of postulates for the foundation of logic*, Annals of Mathematics, Vol. 33 (1932), pp. 346–366 (referred to as Church 1932), Church enunciated five rules of procedure. These were revised by S. C. Kleene, *Proof by cases in formal logic*, Annals of Mathematics, Vol. 35 (1934), pp. 529–544, §1, p. 530 (referred to as Kleene). Whenever we speak of "Church's rules of procedure" we shall mean the revised set.

I is the function such that $(I\mathbf{p})$ is the same as $\mathbf{p}$. I would be $\lambda x \cdot x$ in Church's notation.

J is the function such that $((((J\mathbf{p})\mathbf{q})\mathbf{r})\mathbf{s})$ is the same as $((\mathbf{pq})((\mathbf{ps})\mathbf{r}))$. J would be $\lambda f \lambda x \lambda y \lambda z \cdot f(x, f(z, y))$ in Church's notation.

DEFINITION, by induction, of a *combination*:

1. Any undefined term except (and) is a combination.

2. If $\mathbf{p}$ and $\mathbf{q}$ are combinations, then $(\mathbf{pq})$ is a combination.

Convention. We shall make the convention that a letter in heavy type shall always stand for a combination.

DEFINITION. A left and a right parenthesis *correspond* if the left parenthesis occurs to the left of the right parenthesis and if they, together with the symbols between them, form a combination.

For typographical convenience only we shall omit parenthesis according to the following rule:

1. If a pair of corresponding parentheses lies between another pair of corresponding parentheses, and between the left parenthesis of the inner pair and the left parenthesis of the outer pair only left parentheses occur, then the inner pair may be omitted.

2. If a combination stands alone and its leftmost element is a left parenthesis and its rightmost element is the corresponding right parenthesis, then these two parentheses may be omitted.

Note that the parentheses are not supposed really to be absent. They are actually there, but for convenience we have failed to write them.

A formal definition, like omission of parentheses, is for typographical convenience only. That this is a convenience is seen by the fact that some of our single symbols, such as W, if written in unabbreviated form, would have hundreds of terms in them. T will be defined as JII. This means that whenever we see T written we should understand that what is really there is JII. Therefore if we say "ET, therefore $E(JII)$" we have not used substitution, but have just reminded the reader that ET is actually $E(JII)$.

We shall use Church's method for denoting definitions (see Church 1932, p. 355) and shall list the following, giving on the right the equivalent in Church's notation:

$T \rightarrow JII$	$\lambda x f \cdot f(x)$	(See footnote 4)
$C \rightarrow JT(JT)(JT)$	$\lambda f x y \cdot f(y, x)$	
$B \rightarrow C(JIC)(JI)$	$\lambda f g x \cdot f(g(x))$	
$W \rightarrow C(C(BC(C(BJT)T))T)$	$\lambda f x \cdot f(x, x)$	
$1 \rightarrow BI$	$\lambda f x \cdot f(x)$	
$\mathbf{p} \times \mathbf{q} \rightarrow B\mathbf{pq}$	$\lambda x \cdot \mathbf{p}(\mathbf{q}(x))$	

[4] $\lambda x f \cdot f(x)$ is an abbreviation of $\lambda x \lambda f \cdot f(x)$.

$\mathbf{pq} \times \mathbf{rs} \rightarrow (\mathbf{pq}) \times (\mathbf{rs})$

$\mathbf{p} \times \mathbf{q} \times \mathbf{r} \rightarrow (\mathbf{p} \times \mathbf{q}) \times \mathbf{r}$

$\Phi \rightarrow B(BW) \times BC \times B(BB) \times B$ $\quad \lambda fghx \cdot f(g(x), h(x))$

$\Psi \rightarrow W \times BC \times B(BB) \times B$ $\quad \lambda fgxy \cdot f(g(x), g(y))$

$S \rightarrow \Phi B1$ $\quad \lambda nfx \cdot f(n(f, x))$

$2 \rightarrow S1$ $\quad \lambda fx \cdot f(f(x))$

$3 \rightarrow S2$ $\quad \lambda fx \cdot f(f(f(x)))$,

and so on.

$0 \rightarrow C(C11)1$ $\quad \lambda f \cdot f(1, 1)$

$\oplus \rightarrow C(C1S)1$ $\quad \lambda n \cdot n(S, 1)$

$E \rightarrow B\Sigma T$ $\quad \lambda x \cdot \Sigma f \cdot f(x)$

$\{=\} \rightarrow \Psi\Pi T$ $\quad \lambda xy \cdot \Pi(\lambda f \cdot f(x), \lambda f \cdot f(y))$ (See footnote 5)

$\mathbf{q}^{\mathbf{p}} \rightarrow (\mathbf{pq})$ $\quad \{\mathbf{p}\}(\mathbf{q})$

$\mathbf{p} = \mathbf{q} \rightarrow \{=\}\mathbf{pq}$ $\quad \Pi(\lambda f \cdot f(\mathbf{p}), \lambda f \cdot f(\mathbf{q}))$

$\mathbf{p} + \mathbf{q} \rightarrow \Phi B\mathbf{pq}$ $\quad \lambda fx \cdot \mathbf{p}(f, \mathbf{q}(f, x))$

$\mathbf{pq} + \mathbf{rs} \rightarrow (\mathbf{pq}) + (\mathbf{rs})$

$\mathbf{p} + \mathbf{q} + \mathbf{r} \rightarrow (\mathbf{p} + \mathbf{q}) + \mathbf{r}$

$\mathbf{p} \times \mathbf{q} + \mathbf{r} \times \mathbf{s} \rightarrow (\mathbf{p} \times \mathbf{q}) + (\mathbf{r} \times \mathbf{s})$

Our rules of procedure are:

R1. If $\mathbf{fq}$ and $\Pi\mathbf{fg}$, then $\mathbf{gq}$.

R2. If $\mathbf{fq}$, then $\Sigma\mathbf{f}$.

R3. If $I\mathbf{q}$, then $\mathbf{q}$.

R4. If $\mathbf{q}$, then $I\mathbf{q}$.

R5. If $\mathbf{f}(I\mathbf{qr})$, then $\mathbf{f}(\mathbf{qr})$.

R6. If $\mathbf{f}(\mathbf{qr})$, then $\mathbf{f}(I\mathbf{qr})$.

R7. If $\mathbf{f}(J\mathbf{qrst})$, then $\mathbf{f}(\mathbf{qr}(\mathbf{qts}))$.

R8. If $\mathbf{f}(\mathbf{qr}(\mathbf{qts}))$, then $\mathbf{f}(J\mathbf{qrst})$.

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[5] The expression $\{=\}$ is considered as a single symbol.

We shall often wish to state that a formula $\mathbf{q}$ is either one of the formulas $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n$ or else can be derived from them by a finite number of applications of the rules of procedure. This shall be written:

$$\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n \vdash \mathbf{q}.$$

We shall give, by induction, a precise definition of this notation.

1. If $\mathbf{q}$ is one of the formulas $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n$, then

$\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n \vdash \mathbf{q}$.

2. R1. If $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n \vdash \mathbf{fq}$, and if $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n \vdash \Pi\mathbf{fg}$, then $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n \vdash \mathbf{gq}$.

R2. If $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n \vdash \mathbf{fq}$, then $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n \vdash \Sigma\mathbf{f}$.

R3. If $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n \vdash I\mathbf{q}$, then $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n \vdash \mathbf{q}$.

R4. If $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n \vdash \mathbf{q}$, then $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n \vdash I\mathbf{q}$.

R5. If $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n \vdash \mathbf{f}(I\mathbf{qr})$, then $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n \vdash \mathbf{f}(\mathbf{qr})$.

R6. If $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n \vdash \mathbf{f}(\mathbf{qr})$, then $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n \vdash \mathbf{f}(I\mathbf{qr})$.

R7. If $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n \vdash \mathbf{f}(J\mathbf{qrst})$, then $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n \vdash \mathbf{f}(\mathbf{qr}(\mathbf{qts}))$.

R8. If $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n \vdash \mathbf{f}(\mathbf{qr}(\mathbf{qts}))$, then $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n \vdash \mathbf{f}(J\mathbf{qrst})$.

Our list of postulates is:

P1. $\Pi E(\Phi\Pi(BTI)T)$.

P2. $J = (W \times BC \times B^3C \times B^2B^2)I$.

P3. $B = B^2I \times B$.

P4. $C = B^2I \times C$.

P5. $I = I \times I$.

P6. $CB^3B = BB \times B$.

P7. $CB^3C = BC \times B^2$.

P8. $CB^2W = BW \times B^2$.

P9. $CBI = BI \times I$.

P10. $B \times C = BC \times C \times BB$.

P11. $B \times W = BW \times W \times BC \times B^2B \times B$.

P12. $C^2 = B^2I$.

P13. $BC \times C \times BC = C \times BC \times C$.

P14. $C \times W = BW \times C \times BC$.

P15. $W \times C = W$.

P16. $W^2 = W \times BW$.

Clearly, P1, P2, $\cdots$, P16 $\vdash \mathbf{q}$ means that $\mathbf{q}$ is a consequence of the first sixteen postulates, and hence that $\mathbf{q}$ is a provable formula.

P(m — n), $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_r \vdash \mathbf{q}$ shall be an abbreviation for

Pm, P$\overline{m+1}$, $\cdots$, Pn, $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_r \vdash \mathbf{q}$ and P(m — n) $\vdash \mathbf{q}$ shall be an abbreviation for Pm, P$\overline{m+1}$, $\cdots$, P$n \vdash \mathbf{q}$. Thus P(1–16) $\vdash \mathbf{q}$ means that $\mathbf{q}$ is a consequence of the first sixteen postulates.

Convention. We have numbered the rules of procedure R1 to R8, and the postulates P1 to P16. The methods of proof will be numbered M1, M2, etc. Other intuitive theorems about the system will be numbered T1, T2, etc. Formulas which are consequences of the postulates and are not themselves postulates will be numbered F1, F2, etc.

The notation $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n \vdash \mathbf{q}$ was designed to facilitate the statement of methods of proof. $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n \vdash \mathbf{q}$ is clearly a statement of the fact that there is a step by step method of getting $\mathbf{q}$ from the set of formulas $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n$. To prove the statement $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n \vdash \mathbf{q}$, it is necessary to exhibit the steps by which we can get $\mathbf{q}$ from $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n$.

The following four statements are immediately obvious.

1. If $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n \vdash \mathbf{s}$, and if $\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_m$ be any formulas, and

$$\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_{m+n+k}$$

be any permutation (repetitions allowed) of $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n, \mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_m$, then $\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_{m+n+k} \vdash \mathbf{s}$.

2. If $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n \vdash \mathbf{q}$, and $\mathbf{q} \vdash \mathbf{r}$, then $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n \vdash \mathbf{r}$.
3. If $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n \vdash \mathbf{r}$, and $\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_m, \mathbf{r} \vdash \mathbf{s}$, then

$$\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n, \mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_m \vdash \mathbf{s}.$$

4. If $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n, \mathbf{q}, \mathbf{q} \vdash \mathbf{r}$, then $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n, \mathbf{q} \vdash \mathbf{r}$.

Section B

In this section we prove the important M13, M22, M26, M32, M36, and M37.

M1. a. $JI\mathbf{pqr} \vdash \mathbf{p(rq)}$.
b. $\mathbf{p(rq)} \vdash JI\mathbf{pqr}$.

To prove M1a we must show the steps by which we would get $\mathbf{p(rq)}$ from $JI\mathbf{pqr}$. They are as follows: If $JI\mathbf{pqr}$, $I(JI\mathbf{pqr})$ (R4), $I(I\mathbf{p}(I\mathbf{rq}))$ (R7), $I(\mathbf{p}(I\mathbf{rq}))$ (R5), $\mathbf{p}(I\mathbf{rq})$ (R3), $\mathbf{p(rq)}$ (R5).

We have put after it the rule of procedure by which each step was accomplished. We cannot get $I\mathbf{p}(I\mathbf{rq})$ from $JI\mathbf{pqr}$ by R7 alone, but we can get $I\mathbf{p}(I\mathbf{rq})$ from $JI\mathbf{pqr}$ as follows: If $JI\mathbf{pqr}$, $I(JI\mathbf{pqr})$ (R4), $I(I\mathbf{p}(I\mathbf{rq}))$ (R7), $I\mathbf{p}(I\mathbf{rq})$ (R3).

This use of R3 and R4 in conjunction with R5 to R8 will not be explicitly

mentioned hereafter. Thus we shall write the proof of M1b as follows: If $\mathbf{p}(\mathbf{rq})$, $\mathbf{p}(I\mathbf{rq})$ (R6), $I\mathbf{p}(I\mathbf{rq})$ (R6), $JI\mathbf{pqr}$ (R8).

The proof of M1b is readily seen to consist of the proof of M1a with the steps written in reverse order and with proper changes in the use of the rules of procedure.

M2. If $n \geqq 2$,

a. $\mathbf{f}(I\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3 \cdots \mathbf{p}_n) \vdash \mathbf{f}(\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3 \cdots \mathbf{p}_n)$,

b. $\mathbf{f}(\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3 \cdots \mathbf{p}_n) \vdash \mathbf{f}(I\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3 \cdots \mathbf{p}_n)$.

Proof of a. Use induction with respect to n.

1. When $n = 2$, $\mathbf{f}(I\mathbf{p}_1\mathbf{p}_2) \vdash \mathbf{f}(\mathbf{p}_1\mathbf{p}_2)$ by R5.

2. Assume the theorem true for $n = k \geqq 2$.

If $\mathbf{f}(I\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3 \cdots \mathbf{p}_k\mathbf{p}_{k+1})$, $JI\mathbf{f}\mathbf{p}_{k+1}(I\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3 \cdots \mathbf{p}_k)$	M1b
$JI\mathbf{f}\mathbf{p}_{k+1}(\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3 \cdots \mathbf{p}_k)$	hyp. ind.
$\mathbf{f}(\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3 \cdots \mathbf{p}_k\mathbf{p}_{k+1})$	M1a.

If we recall the rule for omission of parentheses, we see that $\mathbf{f}(I\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3 \cdots \mathbf{p}_k\mathbf{p}_{k+1})$ has the form $\mathbf{f}((I\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3 \cdots \mathbf{p}_k)\mathbf{p}_{k+1})$, and therefore by M1b we can get $JI\mathbf{f}\mathbf{p}_{k+1}(I\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3 \cdots \mathbf{p}_k)$. This has the form $(JI\mathbf{f}\mathbf{p}_{k+1})(I\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3 \cdots \mathbf{p}_k)$ and so we can use the hypothesis of the induction to get $(JI\mathbf{f}\mathbf{p}_{k+1})(\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3 \cdots \mathbf{p}_k)$ or $JI\mathbf{f}\mathbf{p}_{k+1}(\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3 \cdots \mathbf{p}_k)$.

The proof of part b is analogous to that of part a.

M3. If $n \geqq 4$,

a. $\mathbf{f}(J\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4\mathbf{p}_5 \cdots \mathbf{p}_n) \vdash \mathbf{f}(\mathbf{p}_1\mathbf{p}_2(\mathbf{p}_1\mathbf{p}_4\mathbf{p}_3)\mathbf{p}_5 \cdots \mathbf{p}_n)$

b. $\mathbf{f}(\mathbf{p}_1\mathbf{p}_2(\mathbf{p}_1\mathbf{p}_4\mathbf{p}_3)\mathbf{p}_5 \cdots \mathbf{p}_n) \vdash \mathbf{f}(J\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4\mathbf{p}_5 \cdots \mathbf{p}_n)$.

Proof of part a. We use induction with respect to n.

1. When $n = 4$, $\mathbf{f}(J\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4) \vdash \mathbf{f}(\mathbf{p}_1\mathbf{p}_2(\mathbf{p}_1\mathbf{p}_4\mathbf{p}_3))$ by R7.

2. Assume the theorem true for $n = k \geqq 4$.

If $\mathbf{f}(J\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4\mathbf{p}_5 \cdots \mathbf{p}_k\mathbf{p}_{k+1})$, $JI\mathbf{f}\mathbf{p}_{k+1}(J\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4\mathbf{p}_5 \cdots \mathbf{p}_k)$	M1b
$JI\mathbf{f}\mathbf{p}_{k+1}(\mathbf{p}_1\mathbf{p}_2(\mathbf{p}_1\mathbf{p}_4\mathbf{p}_3)\mathbf{p}_5 \cdots \mathbf{p}_k)$	hyp. ind.
$\mathbf{f}(\mathbf{p}_1\mathbf{p}_2(\mathbf{p}_1\mathbf{p}_4\mathbf{p}_3)\mathbf{p}_5 \cdots \mathbf{p}_k\mathbf{p}_{k+1})$	M1a.

M4. a. $T\mathbf{pq} \vdash \mathbf{qp}$

b. $\mathbf{qp} \vdash T\mathbf{pq}$.

Proof. Remembering that T is JII, if $T\mathbf{pq}$, $II(I\mathbf{qp})$ (R7), $I(I\mathbf{qp})$ (R5), $I\mathbf{qp}$ (R3), $\mathbf{qp}$ (R5).

M5. If $n \geqq 3$,

a. $\mathbf{f}(T\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3 \cdots \mathbf{p}_n) \vdash \mathbf{f}(\mathbf{p}_2\mathbf{p}_1\mathbf{p}_3 \cdots \mathbf{p}_n)$

b. $\mathbf{f}(\mathbf{p}_2\mathbf{p}_1\mathbf{p}_3 \cdots \mathbf{p}_n) \vdash \mathbf{f}(T\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3 \cdots \mathbf{p}_n)$.

Proof just like that of M4, as follows:

If $\mathbf{f}(T\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3 \cdots \mathbf{p}_n)$, $\mathbf{f}(II(I\mathbf{p}_2\mathbf{p}_1)\mathbf{p}_3 \cdots \mathbf{p}_n)$	M3a
$\mathbf{f}(I(I\mathbf{p}_2\mathbf{p}_1)\mathbf{p}_3 \cdots \mathbf{p}_n)$	M2a (1)
$\mathbf{f}(I\mathbf{p}_2\mathbf{p}_1\mathbf{p}_3 \cdots \mathbf{p}_n)$	M2a (2)
$\mathbf{f}(\mathbf{p}_2\mathbf{p}_1\mathbf{p}_3 \cdots \mathbf{p}_n)$	M2a.

The hypothesis $n \geqq 3$ is needed to pass from (1) to (2).

M6. a. $C\mathbf{pqr} \vdash \mathbf{prq}$.

b. $\mathbf{prq} \vdash C\mathbf{pqr}$.

Proof. Remembering that C is $JT(JT)(JT)$

If $C\mathbf{pqr}$, $T\mathbf{q}(T\mathbf{rp})$	M3a and M5a.	(1)
$T\mathbf{rpq}$	M4a	(2)
$\mathbf{prq}$	M5a.	

M7. If $n \geqq 4$,

a. $\mathbf{f}(C\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4 \cdots \mathbf{p}_n) \vdash \mathbf{f}(\mathbf{p}_1\mathbf{p}_3\mathbf{p}_2\mathbf{p}_4 \cdots \mathbf{p}_n)$

b. $\mathbf{f}(\mathbf{p}_1\mathbf{p}_3\mathbf{p}_2\mathbf{p}_4 \cdots \mathbf{p}_n) \vdash \mathbf{f}(C\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4 \cdots \mathbf{p}_n)$.

The proof will use a sequence of steps exactly similar to those in the proof of M6. The hypothesis $n \geqq 4$ is used in going from (1) to (2).

M8. a. $B\mathbf{pqr} \vdash \mathbf{p(qr)}$

b. $\mathbf{p(qr)} \vdash B\mathbf{pqr}$.

Proof. B is $C(JIC)(JI)$.

If $B\mathbf{pqr}$, $C(I(JI)\mathbf{p})\mathbf{qr}$	M7a, M3a, and M2a	(1)
$I(JI)\mathbf{prq}$	M6a	(2)
$JI\mathbf{prq}$	M2a	
$\mathbf{p(qr)}$	M1a.	

M9. If $n \geqq 4$,

a. $\mathbf{f}(B\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4 \cdots \mathbf{p}_n) \vdash \mathbf{f}(\mathbf{p}_1(\mathbf{p}_2\mathbf{p}_3)\mathbf{p}_4 \cdots \mathbf{p}_n)$

b. $\mathbf{f}(\mathbf{p}_1(\mathbf{p}_2\mathbf{p}_3)\mathbf{p}_4 \cdots \mathbf{p}_n) \vdash \mathbf{f}(B\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4 \cdots \mathbf{p}_n)$.

Proof by induction with respect to n.

1. When $n = 4$. We can get from $\mathbf{f}(B\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4)$ to $\mathbf{f}(JI\mathbf{p}_1\mathbf{p}_3\mathbf{p}_2\mathbf{p}_4)$ by a sequence of steps similar to those used in M8, using the hypothesis $n = 4$ to pass from (1) to (2).

If $\mathbf{f}(B\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4)$, $\mathbf{f}(JI\mathbf{p}_1\mathbf{p}_3\mathbf{p}_2\mathbf{p}_4)$	
$\mathbf{f}(I\mathbf{p}_1(I\mathbf{p}_2\mathbf{p}_3)\mathbf{p}_4)$	M3a
$\mathbf{f}(\mathbf{p}_1(I\mathbf{p}_2\mathbf{p}_3)\mathbf{p}_4)$	M2a
$JI\mathbf{f}\mathbf{p}_4(\mathbf{p}_1(I\mathbf{p}_2\mathbf{p}_3))$	M1b
$B(JI\mathbf{f}\mathbf{p}_4)\mathbf{p}_1(I\mathbf{p}_2\mathbf{p}_3)$	M8b
$B(JI\mathbf{f}\mathbf{p}_4)\mathbf{p}_1(\mathbf{p}_2\mathbf{p}_3)$	M2a
$JI\mathbf{f}\mathbf{p}_4(\mathbf{p}_1(\mathbf{p}_2\mathbf{p}_3))$	M8a
$\mathbf{f}(\mathbf{p}_1(\mathbf{p}_2\mathbf{p}_3)\mathbf{p}_4)$	M1a.

2. Assume the theorem true for $n = k \geqq 4$.

If $\mathbf{f}(B\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4 \cdots \mathbf{p}_k\mathbf{p}_{k+1})$, $JI\mathbf{f}\mathbf{p}_{k+1}(B\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4 \cdots \mathbf{p}_k)$	M1b
$JI\mathbf{f}\mathbf{p}_{k+1}(\mathbf{p}_1(\mathbf{p}_2\mathbf{p}_3)\mathbf{p}_4 \cdots \mathbf{p}_k)$	hyp. ind.
$\mathbf{f}(\mathbf{p}_1(\mathbf{p}_2\mathbf{p}_3)\mathbf{p}_4 \cdots \mathbf{p}_k\mathbf{p}_{k+1})$	M1a.

M10. a. $W\mathbf{pq} \vdash \mathbf{pqq}$

b. $\mathbf{pqq} \vdash W\mathbf{pq}$

Proof. W is $C(C(BC(C(BJT)T))T)$. Hence if $W\mathbf{pq}$,

$C(BC(C(BJT)T))T\mathbf{qp}$	M6a	
$C(C(BJT)T\mathbf{q})T\mathbf{p}$	M7a and M9a	(1)
$C(BJT)T\mathbf{qp}T$	M6a	(2)
$T\mathbf{q}(T\mathbf{q}T\mathbf{p})$	M7a, M9a, M3a, and M5a	(3)
$T\mathbf{q}T\mathbf{pq}$	M4a	(4)
$\mathbf{pqq}$	M5a.	

M11. If $n \geqq 3$,

a. $\mathbf{f}(W\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3 \cdots \mathbf{p}_n) \vdash \mathbf{f}(\mathbf{p}_1\mathbf{p}_2\mathbf{p}_2\mathbf{p}_3 \cdots \mathbf{p}_n)$

b. $\mathbf{f}(\mathbf{p}_1\mathbf{p}_2\mathbf{p}_2\mathbf{p}_3 \cdots \mathbf{p}_n) \vdash \mathbf{f}(W\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3 \cdots \mathbf{p}_n)$.

Proof by a sequence of steps similar to those used in M10. The hypothesis $n \geqq 3$ is used on the first step, again in going from (1) to (2), and a third time in going from (3) to (4).

M12. a. $\Phi\mathbf{pqrs} \vdash \mathbf{p(qs)(rs)}$

b. $\mathbf{p(qs)(rs)} \vdash \Phi\mathbf{pqrs}$.

Proof is straightforward, using M7a, M8a, M9a, M10a, and the definitions of $\times$ and Φ.

M13. $\mathbf{fp} \vdash E\mathbf{p}$

Proof.

If	$\mathbf{fp}, T\mathbf{pf}$	M4b
	$\Sigma(T\mathbf{p})$	R2
	$B\Sigma T\mathbf{p}$	M8b
	$E\mathbf{p}$	def. of E.

M14. P1, $\mathbf{f}(I\mathbf{p}) \vdash \mathbf{fp}$

Proof.

If	$\mathbf{f}(I\mathbf{p}), B\mathbf{f}I\mathbf{p}$	M8b
	$E\mathbf{p}$	M13
If P1,	$\Phi\Pi(BTI)T\mathbf{p}$	R1 and P1
	$\Pi(BTI\mathbf{p})(T\mathbf{p})$	M12a (1)
If	$\mathbf{f}(I\mathbf{p}), T(I\mathbf{p})\mathbf{f}$	M4b
	$BTI\mathbf{pf}$	M9b
	$T\mathbf{pf}$	R1 and (1)
	$\mathbf{fp}$	M4a.

M15. P1, $\mathbf{f}(T\mathbf{pq}) \vdash \mathbf{f(qp)}$.

Proof similar to that of M5, except that M14 is needed to pass from (1) to (2).

M16. P1, $\mathbf{f}(C\mathbf{pqr}) \vdash \mathbf{f(prq)}$.

Proof similar to that of M6, except that M15 is needed to pass from (1) to (2).

M17. P1, $\mathbf{f}(B\mathbf{pqr}) \vdash \mathbf{f(p(qr))}$.

Proof. By steps similar to those used in M8 and using M16 to pass from (1) to (2), we can go from $\mathbf{f}(B\mathbf{pqr})$ to $\mathbf{f}(JI\mathbf{prq})$.

If	$\mathbf{f}(B\mathbf{pqr}), \mathbf{f}(JI\mathbf{prq})$	
	$\mathbf{f}(I\mathbf{p}(I\mathbf{qr}))$	M3a
	$\mathbf{f(p}(I\mathbf{qr}))$	M2a

$B\mathbf{fp}(I\mathbf{qr})$ M8b

$B\mathbf{fp}(\mathbf{qr})$ M2a

$\mathbf{f}(\mathbf{p}(\mathbf{qr}))$ M8a.

M18. a. $\Psi\mathbf{pqrs} \vdash \mathbf{p}(\mathbf{qr})(\mathbf{qs})$

b. $\mathbf{p}(\mathbf{qr})(\mathbf{qs}) \vdash \Psi\mathbf{pqrs}$.

Proof is straightforward, using M7a, M8a, M9a, M10a, and the definitions of $\times$ and Ψ.

M19. a. $\mathbf{p} = \mathbf{q} \vdash \Pi(T\mathbf{p})(T\mathbf{q})$.

b. $\Pi(T\mathbf{p})(T\mathbf{q}) \vdash \mathbf{p} = \mathbf{q}$.

Proof. $\mathbf{p} = \mathbf{q}$ is $\{=\}\mathbf{pq}$, which is $\Psi\Pi T\mathbf{pq}$.
If $\mathbf{p} = \mathbf{q}$, $\Psi\Pi T\mathbf{pq}$

$\Pi(T\mathbf{p})(T\mathbf{q})$ M18a.

M20. $\mathbf{fp}, \mathbf{p} = \mathbf{q} \vdash \mathbf{fq}$.

Proof. If $\mathbf{fp}$ and $\mathbf{p} = \mathbf{q}$, $T\mathbf{pf}$ and $\Pi(T\mathbf{p})(T\mathbf{q})$ M4b and M19a

$T\mathbf{qf}$ R1

$\mathbf{fq}$ M4a.

M21. P1, $E\mathbf{p} \vdash I\mathbf{p} = \mathbf{p}$.

Proof. If $E\mathbf{p}$ and P1, $\Phi\Pi(BTI)T\mathbf{p}$ R1 and P1

$\Pi(BTI\mathbf{p})(T\mathbf{p})$ M12a

$C\Pi(T\mathbf{p})(BTI\mathbf{p})$ M6b

$C\Pi(T\mathbf{p})(T(I\mathbf{p}))$ M17

$\Pi(T(I\mathbf{p}))(T\mathbf{p})$ M6a

$I\mathbf{p} = \mathbf{p}$ M19b.

M22. P1, $E\mathbf{p} \vdash \mathbf{p} = \mathbf{p}$.

Proof. If P1 and $E\mathbf{p}$, $I\mathbf{p} = \mathbf{p}$ M21

$C\{=\}\mathbf{p}(I\mathbf{p})$ M6b

$C\{=\}\mathbf{pp}$ M14

$\mathbf{p} = \mathbf{p}$ M6a.

M23. P1, $\mathbf{p} = \mathbf{q} \vdash \mathbf{q} = \mathbf{p}$

Proof. If $\mathbf{p} = \mathbf{q}$, $C\{=\}\mathbf{qp}$ M6b

$E\mathbf{p}$ M13

If P1, $\mathbf{p} = \mathbf{p}$ M21

$C\{=\}\mathbf{pp}$ M6b

$C\{=\}\mathbf{pq}$ M20

$\mathbf{q} = \mathbf{p}$ M6a.

M24. $\mathbf{p} = \mathbf{q}, \mathbf{q} = \mathbf{r} \vdash \mathbf{p} = \mathbf{r}$

Proof. If $\mathbf{p} = \mathbf{q}$ and $\mathbf{q} = \mathbf{r}$, $\{=\}\mathbf{pq}$ and $\mathbf{q} = \mathbf{r}$

$\{=\}\mathbf{pr}$ M20.

M25. P1, $E(\mathbf{fp}), \mathbf{p} = \mathbf{q} \vdash \mathbf{fp} = \mathbf{fq}$

Proof. If P1 and $E(\mathbf{fp})$, $\mathbf{fp} = \mathbf{fp}$ M22

$\Psi\{=\}\mathbf{fpp}$ M18b

If $\mathbf{p} = \mathbf{q}$, $\Psi\{=\}\mathbf{fpq}$ M20

$\mathbf{fp} = \mathbf{fq}$ M18a.

M26. a. P1, $E(I\mathbf{p}) \vdash I\mathbf{p} = \mathbf{p}$

b. P1, $E\mathbf{p} \vdash \mathbf{p} = I\mathbf{p}$.

Proof of a. If $E(I\mathbf{p})$, $BEI\mathbf{p}$ M8b

$E\mathbf{p}$ M13

If P1, $I\mathbf{p} = \mathbf{p}$ M21

Proof of b. If P1 and $E\mathbf{p}$, $I\mathbf{p} = \mathbf{p}$ M21

$\mathbf{p} = I\mathbf{p}$ M23

M27. P1, $\mathbf{fp} \vdash \mathbf{f}(I\mathbf{p})$

Proof. If $\mathbf{fp}$, $E\mathbf{p}$ M13

If P1, $\mathbf{p} = I\mathbf{p}$ M26b

$\mathbf{f}(I\mathbf{p})$ M20

M28. P1, $\mathbf{f}(\mathbf{qp}) \vdash \mathbf{f}(T\mathbf{pq})$

Reverse the proof of M15.

M29. P1, $\mathbf{f}(\mathbf{prq}) \vdash \mathbf{f}(C\mathbf{pqr})$

Reverse the proof of M16.

M30. P1, $\mathbf{f}(\mathbf{p}(\mathbf{qr})) \vdash \mathbf{f}(B\mathbf{pqr})$

Reverse the proof of M17.

M31. P1, $E(\mathbf{fp}), \mathbf{f} = \mathbf{g} \vdash \mathbf{fp} = \mathbf{gp}$

Proof. If P1 and $E(\mathbf{fp})$, $E(T\mathbf{pf})$ M28

If $\mathbf{f} = \mathbf{g}$ $T\mathbf{pf} = T\mathbf{pg}$ M25

$\{=\}(T\mathbf{pf})(\mathbf{gp})$ M15

$C\{=\}(\mathbf{gp})(T\mathbf{pf})$ M6b

$C\{=\}(\mathbf{gp})(\mathbf{fp})$ M15

$\mathbf{fp} = \mathbf{gp}$ M6a

M32. a. P1, $E(J\mathbf{pqrs}) \vdash J\mathbf{pqrs} = \mathbf{pq}(\mathbf{psr})$

b. P1, $E(\mathbf{pq}(\mathbf{psr})) \vdash \mathbf{pq}(\mathbf{psr}) = J\mathbf{pqrs}$

Proof. If P1 and $E(J\mathbf{pqrs})$, $\{=\}(J\mathbf{pqrs})(J\mathbf{pqrs})$ M22

$\{=\}(J\mathbf{pqrs})(\mathbf{pq}(\mathbf{psr}))$ M3a

M33. a. P1, $E(C\mathbf{pqr}) \vdash C\mathbf{pqr} = \mathbf{prq}$

b. P1, $E(\mathbf{prq}) \vdash \mathbf{prq} = C\mathbf{pqr}$

Use M22, M16, and M29.

M34. a. P1, $E(B\mathbf{pqr}) \vdash B\mathbf{pqr} = \mathbf{p}(\mathbf{qr})$

b. P1, $E(\mathbf{p}(\mathbf{qr})) \vdash \mathbf{p}(\mathbf{qr}) = B\mathbf{pqr}$

Use M22, M17, and M30.

M35. a. P1, $E(W\mathbf{pq}) \vdash W\mathbf{pq} = \mathbf{pqq}$

b. P1, $E(\mathbf{pqq}) \vdash \mathbf{pqq} = W\mathbf{pq}$

Proof. If P1 and $E(W\mathbf{pq})$, $\{=\}(W\mathbf{pq})(W\mathbf{pq})$ M22.

This is of the form $\mathbf{f}(W\mathbf{pq})$ and therefore we can get $\{=\}(W\mathbf{pq})(\mathbf{pqq})$ by following through successively the steps of M10, using M16 at the first step and to pass from (1) to (2) and using M15 to pass from (3) to (4).

Kleene[6] has shown how to prove certain types of theorems about the parentheses and brackets which occur in well-formed expressions. By exactly similar methods we can prove that:

T1. If $(\mathbf{r}s)$ is the same combination as $(\mathbf{pq})$, then $\mathbf{p}$ is the same as $\mathbf{r}$ and $\mathbf{q}$ is the same as s and therefore s is a combination.

T2. If $\mathbf{p}$ is part of $(\mathbf{qr})$, then either $\mathbf{p}$ is the same as $(\mathbf{qr})$, or else $\mathbf{p}$ is part of $\mathbf{q}$, or else $\mathbf{p}$ is part of $\mathbf{r}$.

M36. If $\mathbf{p}$ is part of $\mathbf{q}$, then $E\mathbf{q} \vdash E\mathbf{p}$.

Proof by induction on $\mathbf{q}$.

1. If $\mathbf{q}$ is an undefined term, and $\mathbf{p}$ is part of $\mathbf{q}$, then $\mathbf{p}$ is $\mathbf{q}$ and the theorem is immediate.

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[6] See Kleene, Section 2, pp. 531–532.

2. Assume the theorem true for $\mathbf{q}_1$ and any $\mathbf{p}$ and for $\mathbf{q}_2$ and any $\mathbf{p}$.
If $\mathbf{p}$ is part of $(\mathbf{q}_1\mathbf{q}_2)$, then by T2:

a. $\mathbf{p}$ is the same as $(\mathbf{q}_1\mathbf{q}_2)$ and the theorem is immediate.

b. $\mathbf{p}$ is part of $\mathbf{q}_1$.

If $E(\mathbf{q}_1\mathbf{q}_2)$, $JIE\mathbf{q}_2\mathbf{q}_1$	M1b
$E\mathbf{q}_1$	M13
$E\mathbf{p}$	hyp. ind.

c. $\mathbf{p}$ is part of $\mathbf{q}_2$.

If $E(\mathbf{q}_1\mathbf{q}_2)$, $BE\mathbf{q}_1\mathbf{q}_2$	M8b
$E\mathbf{q}_2$	M13
$E\mathbf{p}$	hyp. ind.

M37. If $\mathbf{r}$ is part of $\mathbf{p}$, and $\mathbf{q}$ is the result of replacing a single occurrence of $\mathbf{r}$ in $\mathbf{p}$ by $\mathbf{s}$ and leaving the rest of $\mathbf{p}$ unchanged, then P1, $E\mathbf{p}, \mathbf{r} = \mathbf{s} \vdash \mathbf{p} = \mathbf{q}$.
Proof by induction on $\mathbf{p}$.
1. When $\mathbf{p}$ is an undefined term the theorem is obvious.
2. Assume the theorem true for $\mathbf{p}_1$ and any $\mathbf{q}$, $\mathbf{r}$, and $\mathbf{s}$ and for $\mathbf{p}_2$ and any $\mathbf{q}$, $\mathbf{r}$, and $\mathbf{s}$.
If $\mathbf{r}$ is part of $(\mathbf{p}_1\mathbf{p}_2)$, then by T2:
a. $\mathbf{r}$ is $(\mathbf{p}_1\mathbf{p}_2)$ and if $\mathbf{q}$ is the result of replacing $\mathbf{r}$ by $\mathbf{s}$ in $(\mathbf{p}_1\mathbf{p}_2)$, then $\mathbf{r} = \mathbf{s} \vdash (\mathbf{p}_1\mathbf{p}_2) = \mathbf{q}$.
b. $\mathbf{r}$ is part of $\mathbf{p}_1$. Then $\mathbf{q}$ must have the form $(\mathbf{q}_1\mathbf{p}_2)$ where $\mathbf{q}_1$ is the result of a replacement in $\mathbf{p}_1$ of $\mathbf{r}$ by $\mathbf{s}$.

If $E(\mathbf{p}_1\mathbf{p}_2)$, $E\mathbf{p}_1$	M36
If P1 and $\mathbf{r} = \mathbf{s}$, $\mathbf{p}_1 = \mathbf{q}_1$	hyp. ind.
$(\mathbf{p}_1\mathbf{p}_2) = (\mathbf{q}_1\mathbf{p}_2)$	M31

c. $\mathbf{r}$ is part of $\mathbf{p}_2$. Then $\mathbf{q}$ must have the form $(\mathbf{p}_1\mathbf{q}_2)$ where $\mathbf{q}_2$ is the result of a replacement in $\mathbf{p}_2$ of $\mathbf{r}$ by $\mathbf{s}$.

If $E(\mathbf{p}_1\mathbf{p}_2)$, $E\mathbf{p}_2$	M36
If P1 and $\mathbf{r} = \mathbf{s}$, $\mathbf{p}_2 = \mathbf{q}_2$	hyp. ind.
$(\mathbf{p}_1\mathbf{p}_2) = (\mathbf{p}_1\mathbf{q}_2)$	M25.

We find M36 and M37 especially useful in conjunction with M26, M32, M33, M34, and M35. M37 is of course useful in conjunction with postulates P2 to P16 and with any other equalities which can be proved.

Section C

DEFINITION. The occurrences of undefined terms different from (and) in a combination are called the *terms* of the combination.

DEFINITION. The leftmost term of a combination is called the *leading term* of that combination.

It will be noticed that we have nowhere specified that the undefined terms which go to make up a combination must be the same as the undefined terms which we use to build up the particular system of this paper. This leaves the notion of combination quite general, and since combinations occur in the statement of the rules of procedure, the rules of procedure are quite general. Hence the methods of proof M1 to M37 could be applied to many varieties of postulate systems and the properties of combinations which we shall prove will be equally general.

The words "undefined term" are to have their obvious significance. Any symbol which is not defined (such as Π, x, y, Ω) is an undefined term.

DEFINITION. A *proper symbol* is an undefined term different from (,), I, and J.

Convention. The letters **w**, **x**, **y**, and **z**, written lower case and in heavy type shall denote proper symbols.

DEFINITION. A *subcombination* of a given combination is a combination which is a part, or the whole, of the given combination.

DEFINITION, by induction, of the *order* of a subcombination with respect to the combination:

1. A combination is a subcombination of itself of order one.

2. If **r** is a subcombination of order m of **p** and **s** is a subcombination of order n of **q**, then **r** is a subcombination of order m of (**pq**) and **s** is a subcombination of order $n + 1$ of (**pq**).

By use of T1 and T2 it readily follows that the order of a subcombination is unique. Since every term is a subcombination we can speak of the order of a term in a combination.

T3. Every subcombination of a given combination has the same order as its leading term.

Proof follows readily by induction if we assume the theorem true of $\mathbf{q}_1$ and $\mathbf{q}_2$ and prove it of $(\mathbf{q}_1\mathbf{q}_2)$.

T4. If **p** is a subcombination of order m of **q**, and **q** is a subcombination of order n of **r**, then **p** is a subcombination of order $m + n - 1$ of **r**.

Proof by induction simple, if we assume the theorem for $\mathbf{r}_1$ and $\mathbf{r}_2$ and prove it for $(\mathbf{r}_1\mathbf{r}_2)$.

In a combination there is clearly one and only one term of order one, namely the leading term. The terms of order two can be found as follows: Using the method of abbreviation described in Section A, omit as many parentheses as possible. The combination will then appear in the form $\mathbf{p}_0\mathbf{p}_1\mathbf{p}_2 \cdots \mathbf{p}_n$. $\mathbf{p}_0$ is the leading term. $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n$ are each combinations of order two, so their lead-

ing terms are of order two. Furthermore, those are the only terms of order two. The terms of order three would then be the terms which are of order two with respect to $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n$, and so on.

T5. If **p** and **q** are subcombinations of **r** and both have the same order, then either they have the same leading term, or else they have no part in common.

Proof by induction simple if we assume the theorem true for $\mathbf{r}_1$ and $\mathbf{r}_2$ and prove it for $(\mathbf{r}_1\mathbf{r}_2)$.

T6. If $n + m - 1 = l$, and $m \geqq 1$, and $n \geqq 1$, and **p** is a subcombination of order l of **q**, then there is an **r** such that **p** is a subcombination of order m of **r** and **r** is a subcombination of order n of **q**.

Proof by induction on **q**.

1. When **q** is an undefined term the theorem is obvious.

2. Assume the theorem true of $\mathbf{q}_1$ and $\mathbf{q}_2$.

If **p** is part of $(\mathbf{q}_1\mathbf{q}_2)$ then, by T2, we must have one of the three following cases:

a. **p** is all of $(\mathbf{q}_1\mathbf{q}_2)$ and the theorem is obvious.

b. **p** is part of $\mathbf{q}_1$ and the theorem is obvious because the order of **p** with respect to $\mathbf{q}_1$ must be equal to the order of **p** with respect to $(\mathbf{q}_1\mathbf{q}_2)$.

c. **p** is part of $\mathbf{q}_2$. Then l must be at least as great as two. Consider two subcases.

I. $n = 1$. Then $l = m$. Then **p** is of order m in $(\mathbf{q}_1\mathbf{q}_2)$ which is of order one in $(\mathbf{q}_1\mathbf{q}_2)$.

II. $n > 1$. Then $m < l$. Choose **r** so that **p** is of order m in **r** and **r** is of order $n - 1$ in $\mathbf{q}_2$. Then **r** is of order n in $(\mathbf{q}_1\mathbf{q}_2)$.

DEFINITION. If $I\mathbf{q}$ is part of **p**, the act of replacing a single occurrence of $I\mathbf{q}$ in **p** by **q** is called an *I-reduction*. The order of $I\mathbf{q}$ in **p** is called the *order* of the reduction. Either $I\mathbf{q}$ or **q** is called the *part concerned in the reduction* according as we are talking about **p** or the expression that results from the reduction.

DEFINITION. If **q** is part of **p**, the act of replacing a single occurrence of **q** in **p** by $I\mathbf{q}$ is called an *I-expansion*. The order of **q** in **p** is called the *order* of the expansion. Either **q** or $I\mathbf{q}$ is called the *part concerned in the expansion*.

DEFINITION. If $J\mathbf{qrst}$ is part of **p**, the act of replacing a single occurrence of $J\mathbf{qrst}$ in **p** by $\mathbf{qr}(\mathbf{qts})$ is called a *J-reduction*. The order of $J\mathbf{qrst}$ in **p** is called the *order* of the reduction. Either $J\mathbf{qrst}$ or $\mathbf{qr}(\mathbf{qts})$ is called the *part concerned in the reduction*.

DEFINITION. If $\mathbf{qr}(\mathbf{qts})$ is part of **p**, the act of replacing a single occurrence of $\mathbf{qr}(\mathbf{qts})$ in **p** by $J\mathbf{qrst}$ is called a *J-expansion*. The order of $\mathbf{qr}(\mathbf{qts})$ in **p** is called the *order* of the expansion. Either $\mathbf{qr}(\mathbf{qts})$ or $J\mathbf{qrst}$ is called the *part concerned in the expansion*.

When we do not wish to distinguish between *I*- and *J*-reductions we shall speak of a combinatory reduction or merely a reduction; and similarly for expansions.

T7. If **p** goes into **q** by a reduction (or expansion) of order m and **p** is a subcombination of **r** of order n, and **s** differs from **r** only by having a single occurrence of **p** of order n in **r** replaced by **q**, then **r** goes into **s** by a reduction (or expansion) of order $m + n - 1$.

Follows immediately from T4.

T8. If $\mathbf{p}_0\mathbf{p}_1\mathbf{p}_2 \cdots \mathbf{p}_n$ is carried into $\mathbf{q}_0\mathbf{q}_1\mathbf{q}_2 \cdots \mathbf{q}_m$ by a reduction (or expansion) of order $l > 1$, and $\mathbf{p}_0$ is the leading term of $\mathbf{p}_0\mathbf{p}_1\mathbf{p}_2 \cdots \mathbf{p}_n$ and $\mathbf{q}_0$ is the leading term of $\mathbf{q}_0\mathbf{q}_1\mathbf{q}_2 \cdots \mathbf{q}_m$, then $m = n$, and there is an $i \neq 0$ such that $\mathbf{p}_i$ is carried into $\mathbf{q}_i$ by a reduction (or expansion) of order $l - 1$, and, when $j \neq i$, $\mathbf{p}_j$ is the same as $\mathbf{q}_j$.

Proof. $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n$ are each subcombinations of order two of $\mathbf{p}_0\mathbf{p}_1\mathbf{p}_2 \cdots \mathbf{p}_n$, and obviously have different leading terms. Let $\mathbf{s}$ be the part concerned in our reduction (or expansion) of order l. Since $l > 1$, then by T6 we can find an $\mathbf{r}$ such that $\mathbf{s}$ is a subcombination of $\mathbf{r}$ of order $l - 1$ and $\mathbf{r}$ is a subcombination of order two of $\mathbf{p}_0\mathbf{p}_1\mathbf{p}_2 \cdots \mathbf{p}_n$. Then by T5 $\mathbf{r}$ must have the same leading term as some $\mathbf{p}_i$ where $i \neq 0$. Either $\mathbf{r}$ is part of $\mathbf{p}_i$, or $\mathbf{r}$ has elements in common with $\mathbf{p}_{i+1}$. However, if $\mathbf{r}$ has elements in common with $\mathbf{p}_{i+1}$ it has the same leading term as $\mathbf{p}_{i+1}$ (by T5) which is impossible. Hence $\mathbf{r}$ is part of $\mathbf{p}_i$; hence $\mathbf{s}$ is part of $\mathbf{p}_i$. Therefore when the reduction (or expansion) is performed the result is the same as if we performed a reduction (or expansion) or order $l - 1$ on $\mathbf{p}_i$ and left the rest of the $\mathbf{p}_j$'s unchanged.

Suppose that $\mathbf{p}$ goes into $\mathbf{q}$ by a series of reductions and expansions. And suppose that we discover a second series of reductions and expansions which will carry $\mathbf{p}$ into $\mathbf{q}$. If, instead of using the first series to go from $\mathbf{p}$ to $\mathbf{q}$, we use the second series, we shall say that we have replaced the first series by the second.

A series S of n reductions shall be said to be non-overlapping if there is a $\mathbf{p}$, and a set of non-overlapping parts $\mathbf{s}_1, \cdots, \mathbf{s}_n$ of $\mathbf{p}$ such that each $\mathbf{s}_i$ is either $I\mathbf{q}_i$ or $J\mathbf{k}_i\mathbf{l}_i\mathbf{m}_i\mathbf{n}_i$ (let $\bar{\mathbf{s}}_i$ denote $\mathbf{q}_i$ or $\mathbf{k}_i\mathbf{l}_i(\mathbf{k}_i\mathbf{n}_i\mathbf{m}_i)$, resp.), and S consists in replacing successively $\mathbf{s}_1$ by $\bar{\mathbf{s}}_1, \cdots, \mathbf{s}_n$ by $\bar{\mathbf{s}}_n$. By a change in the order of S, we mean the replacement of S by the performance of the substitutions of $\bar{\mathbf{s}}_i$ for $\mathbf{s}_i$ in another order.

T9. Any series of reductions can be replaced by a series of reductions in which every reduction of order n precedes every reduction of order $n + 1$ for all values of n.

LEMMA 1. A series of non-overlapping reductions of orders greater than one followed by a single reduction of order one can be replaced by one or two reductions of order one followed by a set (which may be null) of non-overlapping reductions of orders greater than one.

Proof. Suppose that $\mathbf{p}$ is carried into $\mathbf{q}$ by a series of non-overlapping reductions of orders greater than one followed by a single reduction of order one. Let $\mathbf{p}$ have the form $\mathbf{p}_0\mathbf{p}_1\mathbf{p}_2 \cdots \mathbf{p}_m$ where $\mathbf{p}_0$ is the leading term of $\mathbf{p}$. Then, by T8, the result after making all the reductions of orders greater than one will be $\mathbf{p}_0\mathbf{p}_1'\mathbf{p}_2' \cdots \mathbf{p}_m'$ where $\mathbf{p}_j$ is carried into $\mathbf{p}_j'$ by non-overlapping reductions. Since the reductions were non-overlapping, the order in which they were made is immaterial.

Case 1. Suppose that the reduction of order one is an I-reduction. Then $\mathbf{p}_0$ is I and $\mathbf{q}$ is $\mathbf{p}_1'\mathbf{p}_2' \cdots \mathbf{p}_m'$. Now since the reductions were non-overlapping, it follows that in going from $\mathbf{p}_1$ to $\mathbf{p}_1'$ at most one reduction of order one was made (by T5). Hence we may get from $\mathbf{p}$ to $\mathbf{q}$ as follows:

First go from $\mathfrak{p}$ to $\mathfrak{p}_1\mathfrak{p}_2 \cdots \mathfrak{p}_m$ by an I-reduction of order one. Then we go to $\mathfrak{p}_1'\mathfrak{p}_2 \cdots \mathfrak{p}_m$. If a reduction of order one was made in going from $\mathfrak{p}_1$ to $\mathfrak{p}_1'$, then a reduction of order one will be made in going from $\mathfrak{p}_1\mathfrak{p}_2 \cdots \mathfrak{p}_m$ to $\mathfrak{p}_1'\mathfrak{p}_2 \cdots \mathfrak{p}_m$, and since the reductions are non-overlapping, it can be made first. Then we go from $\mathfrak{p}_1'\mathfrak{p}_2 \cdots \mathfrak{p}_m$ to $\mathfrak{p}_1'\mathfrak{p}_2' \cdots \mathfrak{p}_m'$. It is evident that we have thus gone from $\mathfrak{p}$ to $\mathfrak{q}$ by one or two reductions of order one followed by non-overlapping reductions of higher orders.

Case 2. Suppose that the reduction of order one is a J-reduction. Then $\mathfrak{p}_0$ is J and $\mathfrak{q}$ is $\mathfrak{p}_1'\mathfrak{p}_2'(\mathfrak{p}_1'\mathfrak{p}_4'\mathfrak{p}_3')\mathfrak{p}_5' \cdots \mathfrak{p}_m'$. Hence we may get from $\mathfrak{p}$ to $\mathfrak{q}$ as follows.

First go from $\mathfrak{p}$ to $\mathfrak{p}_1\mathfrak{p}_2(\mathfrak{p}_1\mathfrak{p}_4\mathfrak{p}_3)\mathfrak{p}_5 \cdots \mathfrak{p}_m$ by a J-reduction of order one. Then go to $\mathfrak{p}_1'\mathfrak{p}_2(\mathfrak{p}_1\mathfrak{p}_4\mathfrak{p}_3)\mathfrak{p}_5 \cdots \mathfrak{p}_m$. As in Case 1 this may require a reduction of order one but it can be made first. Then go to $\mathfrak{p}_1'\mathfrak{p}_2'(\mathfrak{p}_1'\mathfrak{p}_4'\mathfrak{p}_3')\mathfrak{p}_5' \cdots \mathfrak{p}_m'$. It is evident that we have thus gone from $\mathfrak{p}$ to $\mathfrak{q}$ by one or two reductions of order one followed by non-overlapping reductions of higher orders.

LEMMA 2. If we have a set of non-overlapping reductions of orders greater than one followed by k reductions of order one, then we can replace them by $k + n$ ($n \geqq 0$) reductions of order one followed by a set (which may be null) of non-overlapping reductions of orders greater than one.

Proof by induction on k.

1. When k is one the theorem follows by Lemma 1.

2. Assume the theorem true for k. Let us have a set of non-overlapping reductions of orders greater than one followed by $k + 1$ reductions of order one. By the hypothesis of the induction we can replace this by $k + n$ ($n \geqq 0$) reductions of order one followed by a set of non-overlapping reductions of orders greater than one and they followed by a single reduction of order one. Then by Lemma 1 we can replace that by $k + n + 1$ or $k + n + 2$ reductions of order one followed by a set of non-overlapping reductions of orders greater than one.

LEMMA 3. If we have a series of k reductions of which m are of order one, we can replace them by $m + n$ ($n \geqq 0$) reductions of order one followed by a series of reductions of orders greater than one.

Proof by induction on k.

1. When k is one the theorem is trivial.

2. Assume the theorem for k. Now suppose that we have $k + 1$ reductions of which m are of order one. If the first reduction is of order one, we use the hypothesis of the induction on the last k reductions. If the first reduction is of order greater than one, we apply the hypothesis of the induction to the last k reductions. We then have a reduction of order greater than one followed by $m + n$ ($n \geqq 0$) reductions of order one and they followed by reductions of orders greater than one. Applying Lemma 2 to the single reduction of order greater than one followed by the $m + n$ reductions of order one, we get $m + n + p$ ($n \geqq 0$, $p \geqq 0$) reductions of order one followed by reductions of orders greater than one.

LEMMA 4. If β is a positive integer, then any series of reductions can be replaced by a series such that, if $n \leqq \beta$ and $m \geqq 1$, every reduction of order n precedes every reduction of order $n + m$.

Proof by induction on β.

1. When $\beta = 1$ the theorem is true by Lemma 3.

2. Assume the theorem true for β. Let us have given a series of reductions. By Lemma 3 we replace them by a series in which all the reductions of order one come first. Now suppose that after making all the reductions of order one we get $\mathbf{p}_0\mathbf{p}_1\mathbf{p}_2 \cdots \mathbf{p}_k$ where $\mathbf{p}_0$ is the leading term. Since now only reductions of orders greater than one will be used the final result will be $\mathbf{p}_0\mathbf{p}_1'\mathbf{p}_2' \cdots \mathbf{p}_k'$ by T8. Let us now replace the original series of reductions by a series in which occur first all the reductions of order one, then all reductions which concern parts of $\mathbf{p}_1$ (evidently reductions which concern parts of $\mathbf{p}_i$ can be interchanged with reductions which concern parts of $\mathbf{p}_j$ when $i \neq j$), then all reductions which concern parts of $\mathbf{p}_2$, $\cdots$, and finally all reductions which concern parts of $\mathbf{p}_k$. Now let us consider the reductions which concern parts of $\mathbf{p}_\alpha$ as reductions of $\mathbf{p}_\alpha$ itself (thereby decreasing the orders of these reductions by one). Then by the hypothesis of the induction we can replace the series of reductions of $\mathbf{p}_\alpha$ by a series in which when $n \leqq \beta$ and $m \geqq 1$, reductions of order n precede reductions of order $n + m$. When we consider these reductions again as reductions of the combination of which $\mathbf{p}_\alpha$ is a part we see that we have arranged the reductions which concern parts of $\mathbf{p}_\alpha$ in such a fashion that when $n \leqq \beta + 1$ and $m \geqq 1$ reductions of order n precede reductions of order $n + m$. Finally we rearrange the reductions as follows: First all reductions of order one, then all reductions of order two which concern parts of $\mathbf{p}_1$, then all reductions of order two which concern parts of $\mathbf{p}_2$, $\cdots$, then all reductions of order two which concern parts of $\mathbf{p}_k$, then all reductions of order three which concern parts of $\mathbf{p}_1$, $\cdots$, then all reductions of order $\beta + 1$ which concern parts of $\mathbf{p}_1$, $\cdots$, then all reductions of order $\beta + 1$ which concern parts of $\mathbf{p}_k$, and finally all the remaining reductions. Thus the lemma is true for $\beta + 1$.

LEMMA 5. The maximum order of any subcombination of $\mathbf{q}$ is less than or equal to the number of terms in $\mathbf{q}$.

Proof by induction immediate.

We are now ready to prove our originally proposed theorem, T9. Consider a $\mathbf{p}$ which is carried into some $\mathbf{q}$ by a series of reductions. Let β be the number of terms in $\mathbf{q}$ and apply Lemma 4. Now by Lemma 5 the last reduction must be of order less than or equal to β and hence by Lemma 4 it must precede all the reductions of orders greater than β. Hence all the reductions are of order β or less and because of Lemma 4 the theorem is proved.

T10. Any series of expansions can be replaced by a series in which the expansions of any order precede those of lower order.

Proof. Since expansions are the reverse of reductions we can use T9.

T11. If $\mathbf{p}$ and $\mathbf{q}$ are subcombinations of $\mathbf{r}$ and the order of $\mathbf{p}$ is less than the order of $\mathbf{q}$, then either $\mathbf{p}$ and $\mathbf{q}$ have no elements in common or else $\mathbf{q}$ is part of $\mathbf{p}$.

Proof by induction on $\mathbf{r}$.

T12. If $\mathbf{p}$ goes into $\mathbf{q}$ by a series of reductions and expansions, then either $\mathbf{p}$ and $\mathbf{q}$ are the same, or else we can carry $\mathbf{p}$ into $\mathbf{q}$ by first a series of reductions and then a series of expansions.

LEMMA 1. If an expansion is followed by a set of non-overlapping reductions, then it can be replaced by a set (which may be null) of non-overlapping reductions followed by a set (which may be null) of expansions.

The part concerned in the expansion may overlap the parts concerned in the reductions. If not, then the expansion may be interchanged with the reductions with no difficulty. If there is overlapping, then three cases arise.

Case 1. One of the reductions overlapping the expansion may have the same order as the expansion. By referring to the definition of reduction and expansion we see that in this case the parts concerned in the reduction and expansion would exactly coincide and hence the expansion could not overlap any more of the reductions. Moreover the reduction would exactly neutralize the effect of the expansion so both could be omitted.

Case 2. One of the reductions overlapping the expansion may be of lower order than the expansion. Then the part concerned in the expansion is part of the part concerned in the reduction (by T11) and therefore the expansion cannot overlap any other reductions. Hence we may make all the other reductions first, then the expansion, and then the reduction which overlaps it. By considering the part concerned in this last reduction we are led to an analysis similar to that given in the proof of Lemma 1, T9. Hence the expansion and reduction can be replaced by a reduction followed by one or two expansions. This reduction will clearly not overlap any of the other reductions.

Case 3. All of the reductions overlapping the expansion are of higher order than the expansion. As in Case 2 we take first the reductions which do not overlap the expansion, then the expansion, then the reductions which do overlap the expansion. We must now consider two cases.

Case I. The expansion is a J-expansion. Then it carries **pq**(**psr**) into J**pqrs**. Since the reductions overlapping the expansion are of higher order than the expansion, the parts concerned with them must be parts of J**pqrs** of at least order two (by T11 and T4). Hence the reductions carry J**pqrs** into J**p′q′r′s′** where **p** goes into **p′** by non-overlapping reductions, **q** goes into **q′** by non-overlapping reductions, **r** goes into **r′** by non-overlapping reductions, and **s** goes into **s′** by non-overlapping reductions. Hence we could carry **pq**(**psr**) into **p′q′**(**p′s′r′**) by a series of non-overlapping reductions (which incidentally would not overlap any of the reductions which had not overlapped the expansion) and then get J**p′q′r′s′** by an expansion.

Case II, where the expansion is an I-expansion, can be handled in a similar fashion.

LEMMA 2. A set of expansions followed by a set of non-overlapping reductions can be replaced by a set (which may be null) of non-overlapping reductions followed by a set (which may be null) of expansions.

Proof like that of Lemma 2, T9.

The theorem now follows by a proof like that of Lemma 3, T9.

DEFINITION. A series of reductions and expansions is said to be a *normal* series or to be in *normal form* if:

1. Each reduction precedes each expansion.
2. Each reduction precedes all reductions of higher order.
3. Each expansion precedes all expansions of lower order.

By T9, T10, and T12 it follows that every series of reductions and expansions can be replaced by a normal series.

T13. If $\mathbf{x}\mathbf{p}_1\mathbf{p}_2 \cdots \mathbf{p}_m$ goes into $\mathbf{y}\mathbf{q}_1\mathbf{q}_2 \cdots \mathbf{q}_n$ by a series of reductions and expansions, then $m = n$, $\mathbf{x}$ is the same as $\mathbf{y}$, and $\mathbf{p}_i$ goes into $\mathbf{q}_i$ by a series of reductions and expansions.

Proof. Replace the series by a normal series. Then $\mathbf{x}\mathbf{p}_1\mathbf{p}_2 \cdots \mathbf{p}_m$ goes into $\mathbf{y}\mathbf{q}_1\mathbf{q}_2 \cdots \mathbf{q}_n$ by a normal series. Since $\mathbf{x}$ is neither I nor J (by our convention $\mathbf{x}$ is a proper symbol) no reductions of order one occur in the series. Also, since $\mathbf{y}$ is not I or J, no expansions of order one occur in the series. Hence the theorem follows by T8.

T14. If $\mathbf{p}$ goes into $\mathbf{q}$ by a series of reductions and expansions and if no reductions can be performed on $\mathbf{r}$ and if $\mathbf{p}$ can be carried into $\mathbf{r}$ by a series of reductions, then only a finite number of successive reductions can be performed on either $\mathbf{p}$ or $\mathbf{q}$. Moreover, if either $\mathbf{p}$ or $\mathbf{q}$ can be carried into $\mathbf{s}$ by a series of reductions, and no reductions can be performed on $\mathbf{s}$, then $\mathbf{r}$ and $\mathbf{s}$ are the same.

LEMMA 1. If no reductions can be performed on $\mathbf{r}$ or $\mathbf{s}$ and both $\mathbf{r}$ and $\mathbf{s}$ can be reached from $\mathbf{p}$ by a series of reductions, then $\mathbf{r}$ and $\mathbf{s}$ are the same.

Proof. Replace the reductions from $\mathbf{p}$ to $\mathbf{r}$ and from $\mathbf{p}$ to $\mathbf{s}$ by normal series. Both normal series will have to consist of all possible reductions of order one, followed by all possible reductions of order two, and so on. Hence the results, $\mathbf{r}$ and $\mathbf{s}$, must be the same.

LEMMA 2. If $\mathbf{p}$ goes into $\mathbf{r}$ by a series of reductions and no reductions are possible on $\mathbf{r}$, then only a finite number of successive reductions are possible on $\mathbf{p}$.

Proof. By going through the proof of T9, we see that if a series of α reductions be replaced by a normal series, the normal series must contain at least α reductions. Now let the normal series of reductions from $\mathbf{p}$ to $\mathbf{r}$ contain β reductions. Then no more than β successive reductions can be performed on $\mathbf{p}$. For, let $\mathbf{p}$ go into $\mathbf{s}$ by α reductions. Then we can go from $\mathbf{r}$ to $\mathbf{s}$ by a series of reductions and expansions. Replace that by a normal series and, since no reductions can be made on $\mathbf{r}$, $\mathbf{r}$ must go into $\mathbf{s}$ by a series of expansions; i.e., $\mathbf{s}$ goes into $\mathbf{r}$ by a series of reductions. Hence $\mathbf{p}$ goes into $\mathbf{r}$ by α or more reductions and hence α is less than or equal to β.

Now let $\mathbf{p}$ go into $\mathbf{q}$ by a series of reductions and expansions and let $\mathbf{p}$ go into $\mathbf{r}$, on which no reductions can be performed, by a series of reductions. Then $\mathbf{r}$ goes into $\mathbf{q}$ by a series of reductions and expansions. Replacing this by a normal series, we see that $\mathbf{r}$ goes into $\mathbf{q}$ by a series of expansions. Hence $\mathbf{q}$ goes into $\mathbf{r}$ by a series of reductions and we can apply Lemma 1 and Lemma 2.

Section D

DEFINITION. The postulates P2 to P16 inclusive are called *conversion postulates*.

DEFINITION. If $\mathbf{r} = \mathbf{s}$ or $\mathbf{s} = \mathbf{r}$ is a conversion postulate and $\mathbf{r}$ is part of $\mathbf{p}$, then the act of replacing a single occurrence of $\mathbf{r}$ in $\mathbf{p}$ by $\mathbf{s}$ and leaving the rest of $\mathbf{p}$ unchanged is called a *p-conversion* (*p* for pustulate).

DEFINITION. A reduction or expansion is called an *r-conversion* (*r* for reduction).

DEFINITION. If $\mathbf{p}$ goes into $\mathbf{q}$ by a series of *p*-conversions we shall say that $\mathbf{p}$ is *p-convertible* into $\mathbf{q}$. If $\mathbf{p}$ goes into $\mathbf{q}$ by a series of reductions and expansions we shall say that $\mathbf{p}$ is *r-convertible* into $\mathbf{q}$.

Note that $\mathbf{p}$ is *r*-convertible into itself as follows: Go from $\mathbf{p}$ to $I\mathbf{p}$ by an expansion and then back to $\mathbf{p}$ by a reduction.

DEFINITION. If $\mathbf{p}$ goes into $\mathbf{q}$ by a series of reductions, then we shall say that $\mathbf{p}$ is *reducible* to $\mathbf{q}$. If the series can be replaced by one in which only reductions of order one occur then we shall say that $\mathbf{p}$ is *reducible in the first sense* to $\mathbf{q}$.

DEFINITION. If $\mathbf{p}$ goes into $\mathbf{q}$ by a series of *p*-conversions or *r*-conversions or both, we say that $\mathbf{p}$ is *convertible* into $\mathbf{q}$, or $\mathbf{p}$ *converts* into $\mathbf{q}$, or that we *convert* $\mathbf{p}$ into $\mathbf{q}$, or that we go from $\mathbf{p}$ to $\mathbf{q}$ by *conversion*. This will be written "$\mathbf{p}$ conv $\mathbf{q}$."

DEFINITION.

Case 1. If the leading term of $\mathbf{p}$ is a proper symbol, the *degree* of $\mathbf{p}$ is 0 and $\mathbf{p}$ is its own *reduced form*.

Case 2. If $\mathbf{p}$ is reducible in the first sense to $\mathbf{q}$ and the leading term of $\mathbf{q}$ is a proper symbol, the *degree* of $\mathbf{p}$ is 0 and $\mathbf{q}$ is the *reduced form* of $\mathbf{p}$.

Case 3. If there is an $m > 0$ and a $\mathbf{q}$ such that:

A. $\mathbf{p}\mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_m$ is reducible in the first sense to $\mathbf{q}$.

B. The leading term of $\mathbf{q}$ is a proper symbol.

C. Either $\mathbf{q}$ does not have the form $\mathbf{r}\mathbf{x}_m$, or $\mathbf{q}$ does have the form $\mathbf{r}\mathbf{x}_m$ but $\mathbf{p}\mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_{m-1}$ is not reducible in the first sense to $\mathbf{r}$; then m is the *degree* of $\mathbf{p}$ and $\mathbf{q}$ is the *reduced form* of $\mathbf{p}\mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_m$.

In all three cases $\mathbf{p}$ is said to be a *proper* combination.

$W\Pi\Sigma$ is a proper combination of degree zero. WW is a proper combination of degree one. WWW is not a proper combination. However $I\Pi(WWW)$ is a proper combination of degree zero.

T15. a. I is a proper combination of degree one.

b. J is a proper combination of degree four.

c. All proper symbols are proper combinations of degree zero.

d. If $\mathbf{r} = \mathbf{s}$ is a conversion postulate, then $\mathbf{r}$ and $\mathbf{s}$ are proper and have the same degree and the same reduced form.

e. If $\mathbf{p}$ is proper and of degree m and $0 \leqq n \leqq m$, then $\mathbf{p}\mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_n$ is proper and of degree $m - n$.

f. If $\mathbf{p}\mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_n$ is proper and of degree m, then $\mathbf{p}$ is proper and of degree less than or equal to $m + n$.

g. If $\mathbf{p}$ is proper and of degree zero, then $\mathbf{p}\mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_k$ is proper and of degree zero.

Proof obvious by referring to the definitions.

T16. If $\mathbf{p}$ conv $\mathbf{q}$, and $\mathbf{p}$ has the degree m, and the reduced form of $\mathbf{p}\mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_m$

is $\mathbf{x}\mathbf{p}_1\mathbf{p}_2 \cdots \mathbf{p}_k$ (we note the possibility that $\mathbf{x}$ might be one of the $\mathbf{x}_i$'s), then $\mathbf{q}$ has the degree m or less, and the reduced form of $\mathbf{q}\mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_m$ is $\mathbf{x}\mathbf{q}_1\mathbf{q}_2 \cdots \mathbf{q}_k$ where $\mathbf{p}_i$ conv $\mathbf{q}_i$.

Proof. We shall first consider the case where $\mathbf{p}$ is convertible into $\mathbf{q}$ by reductions and expansions only. Since $\mathbf{p}$ conv $\mathbf{q}$, $\mathbf{p}\mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_m$ conv $\mathbf{q}\mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_m$. Hence $\mathbf{x}\mathbf{p}_1\mathbf{p}_2 \cdots \mathbf{p}_k$ conv $\mathbf{q}\mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_m$. Consider the normal series of reductions and expansions going from $\mathbf{q}\mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_m$ to $\mathbf{x}\mathbf{p}_1\mathbf{p}_2 \cdots \mathbf{p}_k$. No expansions of order one can occur. Now after applying all the reductions of order one of the normal series to $\mathbf{q}\mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_m$, we get a result which goes into $\mathbf{x}\mathbf{p}_1\mathbf{p}_2 \cdots \mathbf{p}_k$ by reductions and expansions of orders greater than one and which must therefore (by T8) have the form $\mathbf{x}\mathbf{q}_1\mathbf{q}_2 \cdots \mathbf{q}_k$ where $\mathbf{p}_i$ conv $\mathbf{q}_i$ for all i. But $\mathbf{x}$ is neither I nor J, and therefore $\mathbf{q}\mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_m$ is of degree zero and has $\mathbf{x}\mathbf{q}_1\mathbf{q}_2 \cdots \mathbf{q}_k$ for reduced form. Hence $\mathbf{q}$ has degree m or less.

LEMMA. If $\mathbf{p}$ goes into $\mathbf{q}$ by p-conversions only, the theorem is true.

We shall prove the lemma by an induction with respect to the number of reductions of order one required to carry $\mathbf{p}\mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_m$ into $\mathbf{x}\mathbf{p}_1\mathbf{p}_2 \cdots \mathbf{p}_k$. Call this number α.

1. $\alpha = 0$. Then m is zero and $\mathbf{p}$ has the form $\mathbf{x}\mathbf{p}_1\mathbf{p}_2 \cdots \mathbf{p}_k$. In order to prove the lemma for $\alpha = 0$, it is only necessary to show how to prove it for a single p-conversion. Hence, let $\mathbf{r} = \mathbf{s}$ or $\mathbf{s} = \mathbf{r}$ be a conversion postulate, let $\mathbf{r}$ be part of $\mathbf{p}$ and let $\mathbf{q}$ be the result of replacing a single occurrence of $\mathbf{r}$ in $\mathbf{p}$ by $\mathbf{s}$. By inspecting the conversion postulates we see that the leading term of $\mathbf{r}$ is I or J and hence cannot be $\mathbf{x}$. Therefore we can reason as in the proof of T8 and conclude that $\mathbf{q}$ has the form $\mathbf{x}\mathbf{p}_1\mathbf{p}_2 \cdots \mathbf{p}'_l \cdots \mathbf{p}_k$ where $\mathbf{p}_l$ conv $\mathbf{p}'_l$. Hence $\mathbf{q}$ is of degree zero.

2. Assume the lemma true for α. Let $\mathbf{p}\mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_m$ go into $\mathbf{x}\mathbf{p}_1\mathbf{p}_2 \cdots \mathbf{p}_k$ by $\alpha + 1$ reductions of order one. Again we merely prove the lemma for the case that $\mathbf{p}$ goes into $\mathbf{q}$ by a single p-conversion. Hence let $\mathbf{r} = \mathbf{s}$ or $\mathbf{s} = \mathbf{r}$ be a conversion postulate, let $\mathbf{r}$ be part of $\mathbf{p}$ and let us get $\mathbf{q}$ by replacing a single occurrence of $\mathbf{r}$ in $\mathbf{p}$ by $\mathbf{s}$. Let $\mathbf{p}$ have the form $\mathbf{p}'_0\mathbf{p}'_1 \cdots \mathbf{p}'_l$ where $\mathbf{p}'_0$ is the leading term.

Case 1. $\mathbf{p}'_0$ is not the leading term of $\mathbf{r}$. Reasoning as in the proof of T8, we conclude that $\mathbf{q}$ has the form $\mathbf{p}'_0\mathbf{p}'_1 \cdots \mathbf{p}''_j \cdots \mathbf{p}'_l$ where $\mathbf{p}'_j$ conv $\mathbf{p}''_j$. Now since $\mathbf{p}'_0$ is the leading term of both $\mathbf{p}$ and $\mathbf{q}$, we can perform the same reduction of order one on $\mathbf{p}\mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_m$ and $\mathbf{q}\mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_m$, getting respectively $\mathbf{p}'$ and $\mathbf{q}'$. $\mathbf{p}'$ clearly goes into $\mathbf{q}'$ by p-conversions only and only α reductions of order one are needed to carry $\mathbf{p}'$ into $\mathbf{x}\mathbf{p}_1\mathbf{p}_2 \cdots \mathbf{p}_k$. Hence the degree of $\mathbf{q}'$ is zero and $\mathbf{q}'$ has $\mathbf{x}\mathbf{q}_1\mathbf{q}_2 \cdots \mathbf{q}_k$ for reduced form. But $\mathbf{q}\mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_m$ goes into $\mathbf{q}'$ by a single reduction of order one and hence $\mathbf{q}$ is of degree m or less and $\mathbf{q}\mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_m$ has $\mathbf{x}\mathbf{q}_1\mathbf{q}_2 \cdots \mathbf{q}_k$ for reduced form.

Case 2. $\mathbf{p}'_0$ is the leading term of $\mathbf{r}$. Then there is a j such that $\mathbf{r}$ is $\mathbf{p}'_0\mathbf{p}'_1 \cdots \mathbf{p}'_j$. Hence $\mathbf{p}$ is $\mathbf{r}\mathbf{p}'_{j+1}\mathbf{p}'_{j+2} \cdots \mathbf{p}'_l$ and $\mathbf{q}$ is $\mathbf{s}\mathbf{p}'_{j+1}\mathbf{p}'_{j+2} \cdots \mathbf{p}'_l$. Now by inspecting the conversion postulates one by one, we see that, unless the degree of $\mathbf{r}$ and $\mathbf{s}$ is less than or equal to $l - j + m$, $\mathbf{r}\mathbf{p}'_{j+1}\mathbf{p}'_{j+2} \cdots \mathbf{p}'_l\mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_m$ cannot be reduced

to a form where the leading term is not I or J. But, by hypothesis $\mathbf{p}\mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_m$ or $\mathbf{r}\mathbf{p}'_{j+1}\mathbf{p}'_{j+2} \cdots \mathbf{p}'_l\mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_m$ can be reduced to such a form and by reductions of order one only. Therefore the degree of $\mathbf{r}$ and $\mathbf{s}$ is less than or equal to $l - j + m$. By again inspecting the conversion postulates one by one we see that $\mathbf{r}\mathbf{y}_1\mathbf{y}_2 \cdots \mathbf{y}_{l-j+m}$ can be carried by reductions and expansions only into $\mathbf{s}\mathbf{y}_1\mathbf{y}_2 \cdots \mathbf{y}_{l-j+m}$. Therefore we can go from $\mathbf{r}\mathbf{p}'_{j+1}\mathbf{p}'_{j+2} \cdots \mathbf{p}'_l\mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_m$ to $\mathbf{s}\mathbf{p}'_{j+1}\mathbf{p}'_{j+2} \cdots \mathbf{p}'_l\mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_m$ by reductions and expansions only. Hence $\mathbf{q}\mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_m$ is of degree zero and has $\mathbf{x}\mathbf{q}_1\mathbf{q}_2 \cdots \mathbf{q}_k$ for reduced form. Hence $\mathbf{q}$ is of degree m or less.

By use of the Lemma, the theorem is readily proved.

COROLLARY 1. If $\mathbf{p}$ has the degree m and $\mathbf{p}$ conv $\mathbf{q}$, then $\mathbf{q}$ has the degree m.

Proof obvious since $\mathbf{q}$ conv $\mathbf{p}$ if $\mathbf{p}$ conv $\mathbf{q}$.

COROLLARY 2. If $\mathbf{x}\mathbf{p}_1\mathbf{p}_2 \cdots \mathbf{p}_m$ conv $\mathbf{y}\mathbf{q}_1\mathbf{q}_2 \cdots \mathbf{q}_n$, then $m = n$ and $\mathbf{x}$ and $\mathbf{y}$ are the same and $\mathbf{p}_i$ conv $\mathbf{q}_i$ for all i.

T17. If $\mathbf{p}$ conv $\mathbf{q}$, and $\mathbf{p}$ is of degree m, and $\mathbf{p}\mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_m$ can be reduced to $\mathbf{r}$, and $\mathbf{r}$ does not contain I or J, then $\mathbf{q}\mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_m$ can be reduced to $\mathbf{r}$. Moreover if $\mathbf{q}\mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_m$ can be reduced to $\mathbf{s}$ and $\mathbf{s}$ does not contain I or J, then $\mathbf{r}$ and $\mathbf{s}$ are the same.

Proof. Since $\mathbf{r}$ does not contain I or J it can evidently be obtained from $\mathbf{p}\mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_m$ by first making all possible reductions of order one, then all possible reductions of order two, and so on. We wish to show that a similar set of reductions applied to $\mathbf{q}\mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_m$ will give $\mathbf{r}$.

Suppose that after applying all possible reductions of order one to $\mathbf{p}\mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_m$ we get $\mathbf{x}\mathbf{p}_1\mathbf{p}_2 \cdots \mathbf{p}_k$ (the $\mathbf{x}$ may be one of the $\mathbf{x}_i$'s). Then by T16 there will be possible only a finite number of reductions of order one that can be applied to $\mathbf{q}\mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_m$ and the result will be $\mathbf{x}\mathbf{q}_1\mathbf{q}_2 \cdots \mathbf{q}_k$ where $\mathbf{p}_i$ conv $\mathbf{q}_i$. Now make all possible reductions of order one of $\mathbf{p}_1$ and call the result $\mathbf{p}'_0\mathbf{p}'_1 \cdots \mathbf{p}'_l$ where $\mathbf{p}'_0$ is the leading term. Then $\mathbf{p}'_0$ must be different from I or J since these do not occur in $\mathbf{r}$. Hence $\mathbf{p}_1$ is of degree zero and hence $\mathbf{q}_1$ is of degree zero. Hence upon making all possible reductions of order one of $\mathbf{q}_1$ we would get (by T16) $\mathbf{p}'_0\mathbf{q}'_1\mathbf{q}'_2 \cdots \mathbf{q}'_l$ where $\mathbf{p}'_i$ conv $\mathbf{q}'_i$. Keeping up this process to finer and finer subdivisions we finally get the same result by both series of reductions.

T17 is number 1 of Curry's *Hauptsätze*, mentioned on page 528 and proved on page 799 of Curry 1930.

MC. If $\mathbf{p}$ conv $\mathbf{q}$, then P(1–16), $E\mathbf{p} \vdash \mathbf{p} = \mathbf{q}$.

Proof. Use M26, M32, M36, and M37.

T18. If P(1–16) $\vdash \mathbf{p} = \mathbf{q}$, then $\mathbf{p}$ conv $\mathbf{q}$.

Proof. We remember that the formula $\Pi E(\Phi\Pi(BTI)T)$ is P1.

LEMMA. If P(1–16) $\vdash \mathbf{p}$, and if $\mathbf{p}$ is convertible into $\Pi\mathbf{f}\mathbf{g}$, then either $\Pi\mathbf{f}\mathbf{g}$ is convertible into P1 or else $\mathbf{f}$ conv $\mathbf{g}$.

Let us call the applications of the rules of procedure which we make in deriving $\mathbf{p}$, steps in the proof of $\mathbf{p}$. Then we prove the lemma by an induction on the number of steps used to prove $\mathbf{p}$.

1. The postulates are the only formulas which can be proved in zero steps.

The lemma is readily seen to be true for the first sixteen postulates because, if $\mathbf{r} = \mathbf{s}$ is a conversion postulate, it is convertible into $\Pi(T\mathbf{r})(T\mathbf{s})$ and hence if it is convertible into $\Pi\mathbf{fg}$ then by Corollary 2, T16, $\mathbf{f}$ conv $T\mathbf{r}$ and $\mathbf{g}$ conv $T\mathbf{s}$. But $T\mathbf{r}$ conv $T\mathbf{s}$.

2. Assume the lemma true for k steps. Now let us consider $\mathbf{p}$ proved in $k+1$ steps. If the last step was an application of one of rules of procedure R3 to R8 inclusive, then $\mathbf{p}$ was derived from $\mathbf{q}$ by one of the rules R3 to R8 and $\mathbf{q}$ was proved in k steps. Hence the lemma is true for $\mathbf{q}$. However if one of the rules R3 to R8 was used to go from $\mathbf{q}$ to $\mathbf{p}$, then $\mathbf{q}$ conv $\mathbf{p}$ and hence the lemma is true for $\mathbf{p}$.

If the last step was an application of rule R2, then $\mathbf{p}$ is of the form $\Sigma\mathbf{h}$ and hence (by Corollary 2, T16) cannot be converted into $\Pi\mathbf{fg}$.

If the last step was an application of R1, then we have $\mathbf{fq}$ and $\Pi\mathbf{fg}$ both proved in k steps or less and $\mathbf{p}$ is $\mathbf{gq}$. By the hypothesis of our lemma, either $\Pi\mathbf{fg}$ is convertible into P1 or else $\mathbf{f}$ conv $\mathbf{g}$.

Case 1. $\Pi\mathbf{fg}$ conv P1. Then by Corollary 2, T16, $\mathbf{f}$ conv E and $\mathbf{g}$ conv $\Phi\Pi(BTI)T$. Hence $\mathbf{p}$, which is $\mathbf{gq}$, is convertible into $\Phi\Pi(BTI)T\mathbf{q}$, which is convertible into $\Pi(T\mathbf{q})(T\mathbf{q})$. Hence if $\mathbf{p}$ conv $\Pi\mathbf{f'g'}$, then $\Pi\mathbf{f'g'}$ conv $\Pi(T\mathbf{q})(T\mathbf{q})$, hence by Corollary 2, T16, $\mathbf{f'}$ conv $T\mathbf{q}$ and $\mathbf{g'}$ conv $T\mathbf{q}$.

Case 2. $\mathbf{f}$ conv $\mathbf{g}$. Then $\mathbf{p}$, which is $\mathbf{gq}$, is convertible into $\mathbf{fq}$. But $\mathbf{fq}$ was proved in k steps or less and so the lemma is true of $\mathbf{fq}$ and hence of $\mathbf{gq}$ or $\mathbf{p}$.

Proof of theorem. Assume that $\mathbf{p} = \mathbf{q}$ has been proved. It is convertible into $\Pi(T\mathbf{p})(T\mathbf{q})$, and so by the lemma, either $\Pi(T\mathbf{p})(T\mathbf{q})$ conv P1 or else $T\mathbf{p}$ conv $T\mathbf{q}$. However $\Pi(T\mathbf{p})(T\mathbf{q})$ is not convertible into P1. For if it were we would have E conv $T\mathbf{p}$, and hence $T\mathbf{p}\Pi$ conv $E\Pi$. However this is not the case (by Corollary 2, T16) for $T\mathbf{p}\Pi$ conv $\Pi\mathbf{p}$ and $E\Pi$ conv $\Sigma(T\Pi)$. Hence $T\mathbf{p}$ conv $T\mathbf{q}$. Hence $T\mathbf{p}I$ conv $T\mathbf{q}I$. But $\mathbf{p}$ conv $T\mathbf{p}I$ and $\mathbf{q}$ conv $T\mathbf{q}I$.

We have now proved that our system is not too strong, i.e. that we cannot prove certain undesirable theorems in it. In a second part, to follow soon, we shall prove that it is strong enough.

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A NOTE ON THE POINCARÉ THEOREM ON IMPURE RIEMANN MATRICES[1]

BY A. ADRIAN ALBERT

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1. A p by $2p$ complex matrix ω is a *Riemann matrix* if there exists a rational $2p$-rowed skew-symmetric matrix C such that $\omega C\omega' = 0$, $i\omega C\bar{\omega}'$ is positive definite. If α is p-rowed complex and non-singular, A is $2p$-rowed rational and non-singular, then $\alpha\omega A$ and ω are said to be isomorphic.

The period matrix of the abelian integrals of the first kind on a Riemann surface is a Riemann matrix. It was in this connection that Poincaré proved[2] that if ω is impure, that is ω is isomorphic to

$$\begin{pmatrix} \omega_1 & 0 \\ \omega_3 & \omega_2 \end{pmatrix},$$

where ω_1 is a Riemann matrix, then ω is also isomorphic to

$$\begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix}$$

where ω_2 is also a Riemann matrix.

The above theorem was generalized to the case where ω is any Riemann matrix by G. Scorza. Scorza used a rather complicated projective geometric method and his proof (as well as all previous proofs) is not very simple. I shall give here an elementary matrix proof of the above theorem and the later Scorza reduction of any impure Riemann matrix to its pure components.

2. If ω is a Riemann matrix with C as principal matrix, then $\omega C\omega' = 0$ and $i\omega C\bar{\omega}'$ is positive definite. Hence $\gamma = \omega C\bar{\omega}'$ is non-singular, and

$$\Omega \equiv \begin{pmatrix} \omega \\ \bar{\omega} \end{pmatrix}, \qquad \Omega C\Omega' = \begin{pmatrix} 0 & \gamma \\ \bar{\gamma} & 0 \end{pmatrix}, \tag{1}$$

where $\bar{\omega}C\bar{\omega}' = 0$, $\bar{\gamma} = \bar{\omega}C\omega'$. Then $\Omega C\Omega'$ is a non-singular product of the $2p$-rowed square matrices Ω, C, Ω'. Hence Ω and C are non-singular,

$$(\Omega C\Omega')^{-1} = \begin{pmatrix} 0 & \bar{\gamma}^{-1} \\ \gamma^{-1} & 0 \end{pmatrix}, \qquad (C\Omega')^{-1} = \begin{pmatrix} 0 & \bar{\gamma}^{-1} \\ \gamma^{-1} & 0 \end{pmatrix} \Omega . \tag{2}$$

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[1] Presented to the Society March 30, 1934. This paper was presented in my seminar on Riemann matrices at the Institute for Advanced Study in February 1934, and is the paper to which H. Weyl refers in these Annals vol. 35 (1934), pp. 714–729.

[2] For detailed references, see S. Lefschetz's report, Chapters XV–XVII, of the Report of the Committee on Rational Transformations, Bulletin of the National Research Council, 63 (1928), pp. 310–392.

We now prove

LEMMA 1. *Let $\tau C\omega' = 0$ for a complex matrix τ. Then $\tau = \delta\omega$ where δ is a complex matrix.*

For also $\bar{\tau}C\bar{\omega}' = 0$ and

$$\begin{pmatrix} \tau \\ \bar{\tau} \end{pmatrix} C\Omega' = \begin{pmatrix} \tau C \\ \bar{\tau}C \end{pmatrix} (\omega' \ \bar{\omega}') = \begin{pmatrix} 0 & \tau C\bar{\omega}' \\ \bar{\tau}C\omega' & 0 \end{pmatrix}, \tag{3}$$

from which

$$\begin{pmatrix} \tau \\ \bar{\tau} \end{pmatrix} = \begin{pmatrix} 0 & \tau C\bar{\omega}' \\ \bar{\tau}C\omega' & 0 \end{pmatrix} \begin{pmatrix} 0 & \bar{\gamma}^{-1} \\ \gamma^{-1} & 0 \end{pmatrix} \Omega = \begin{pmatrix} \delta & 0 \\ 0 & \bar{\delta} \end{pmatrix} \Omega = \begin{pmatrix} \delta & \omega \\ \bar{\delta} & \bar{\omega} \end{pmatrix}, \tag{4}$$

where $\delta = \tau C\bar{\omega}'\gamma^{-1}$. Then $\tau = \delta\omega$.

We also have the trivial

LEMMA 2. *A Riemann matrix*

$$\omega = \begin{pmatrix} \omega_1 & \omega_3 \\ 0 & \omega_2 \end{pmatrix}, \tag{5}$$

where ω has q-rows and $2q$-columns, is isomorphic to[3]

$$\begin{pmatrix} 0 & I(p-q) \\ I(q) & 0 \end{pmatrix} \begin{pmatrix} \omega_1 & \omega_3 \\ 0 & \omega_2 \end{pmatrix} \begin{pmatrix} 0 & I(2q) \\ I(2p-2q) & 0 \end{pmatrix} = \begin{pmatrix} \omega_2 & 0 \\ \omega_3 & \omega_1 \end{pmatrix}. \tag{6}$$

We now prove the Poincaré theorem

THEOREM 1. *Let ω be a Riemann matrix of genus p,*

$$\omega = \begin{pmatrix} \omega_1 & 0 \\ \omega_3 & \omega_2 \end{pmatrix}, \tag{7}$$

where ω_1 has q rows and $2q$ columns and $q < p$. Then ω_1 and ω_2 are Riemann matrices and ω is isomorphic to

$$\begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix}. \tag{8}$$

For let C be a principal matrix of ω and write

$$C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}, \tag{9}$$

where C_1 has $2q$ rows and columns. Then

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[3] We use the notation $I(r)$ for an r-rowed identity matrix.

$$\omega C\omega' = \begin{pmatrix} \omega_1 & 0 \\ \omega_3 & \omega_2 \end{pmatrix}\begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}\omega' = \begin{pmatrix} \omega_1 C_1 & \omega_1 C_2 \\ \omega_3 C_1 + \omega_2 C_3 & \omega_3 C_2 + \omega_2 C_4 \end{pmatrix}\begin{pmatrix} \omega_1' & \omega_3' \\ 0 & \omega_2' \end{pmatrix}$$

(10)

$$= \begin{pmatrix} \omega_1 C_1 \omega_1' & \omega_1 [C_1 \omega_3' + C_2 \omega_2'] \\ [\omega_3 C_1 + \omega_2 C_3]\, \omega_1' & [\omega_3 C_1 + \omega_2 C_3]\, \omega_3' + [\omega_3 C_2 + \omega_2 C_4]\, \omega_2' \end{pmatrix},$$

while similarly

$$(11)\quad i\omega C\bar{\omega}' = \begin{pmatrix} i\omega_1 C_1 \bar{\omega}_1' & i\omega_1 [C_1 \bar{\omega}_3' + C_2 \bar{\omega}_2'] \\ i\,[\omega_3 C_1 + \omega_2 C_3]\, \bar{\omega}_1' & i\,[\omega_3 C_1 + \omega_2 C_3]\, \bar{\omega}_3' + i\,[\omega_3 C_2 + \omega_2 C_4]\, \bar{\omega}_2' \end{pmatrix}.$$

The matrix $\omega C\omega' = 0$ so that we have

$$(12)\qquad \omega_1 C_1 \omega_1' = 0\,, \qquad [\omega_3 C_1 + \omega_2 C_3]\, \omega_1' = 0\,.$$

Also $i\omega C\bar{\omega}'$ is positive definite and *its principal minor* $i\omega_1 C_1 \bar{\omega}_1'$ *must be positive definite.* Thus ω_1 is a Riemann matrix with principal matrix C_1 and C_1 is non-singular. We define

$$(13)\qquad \tau = \omega_3 + \omega_2 C_3 C_1^{-1},$$

and (10) implies that $\tau C_1 \omega_1' = 0$. By Lemma 1 we obtain $\tau = \delta\omega_1$.

Define

$$(14)\qquad \alpha = \begin{pmatrix} I(q) & 0 \\ -\delta & I(p-q) \end{pmatrix}, \qquad A = \begin{pmatrix} I(2q) & 0 \\ C_3 C_1^{-1} & I(2p-2q) \end{pmatrix}$$

whence α and A are non-singular and A is rational. Then ω is isomorphic to

$$\alpha\omega A = \alpha\begin{pmatrix} \omega_1 & 0 \\ \omega_3 & \omega_2 \end{pmatrix}\begin{pmatrix} I(2q) & 0 \\ C_3 C_1^{-1} & I(2p-2q) \end{pmatrix} = \alpha\begin{pmatrix} \omega_1 & 0 \\ \tau & \omega_2 \end{pmatrix}$$

(15)

$$= \begin{pmatrix} I(q) & 0 \\ -\delta & I(p-q) \end{pmatrix}\begin{pmatrix} \omega_1 & 0 \\ \delta\omega_1 & \omega_2 \end{pmatrix} = \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix}.$$

The matrix ω_1 is a Riemann matrix by the above proof. By Lemma 2 the matrix ω is isomorphic to

$$(16)\qquad \begin{pmatrix} \omega_2 & 0 \\ 0 & \omega_1 \end{pmatrix}$$

and ω_2 is a Riemann matrix by the above proof.

As an immediate corollary of Lemma 2 and Theorem 1, we have

THEOREM 2. *Let ω have the form*

$$\omega = \begin{pmatrix} \omega_1 & \omega_3 \\ 0 & \omega_2 \end{pmatrix} \tag{17}$$

where ω_1 has q-rows and $2q$ columns and q is less than the genus p of ω. Then ω is impure and is isomorphic to

$$\begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix} \tag{18}$$

where ω_1 and ω_2 are Riemann matrices.

3. Let now ω be any Riemann matrix. If ω is not pure, then ω is isomorphic to a matrix of the form (8). Applying the same criterion to ω_1 and ω_2 we ultimately obtain a complete reduction of ω. In this final form we have ω isomorphic to

$$\omega_0 = \begin{bmatrix} \Lambda_1 & & & 0 \\ & \Lambda_2 & & \\ & & \ddots & \\ 0 & & & \Lambda_f \end{bmatrix} \tag{19}$$

where

$$\Lambda_j = \begin{bmatrix} \omega_{j1} & & 0 \\ & \ddots & \\ 0 & & \omega_{jt_j} \end{bmatrix}, \ \omega_{jk} = \omega_{j1} \qquad (k = 1, \cdots, t_j), \tag{20}$$

and ω_{j1} is a pure Riemann matrix not isomorphic to ω_{k1} for $j \neq k$. We may obviously have $t_j = 1$ or $f = 1$ in a special case.

We shall now prove an analogue of Schur's Lemma of the theory of group representations.

THEOREM 3. *Let ω_1 and ω_2 be pure Riemann matrices of genus p_1, p_2 respectively, and let*

$$\alpha\omega_1 = \omega_2 A$$

for a rational $2p_2$ by $2p_1$ matrix A and a complex p_2 by p_1 matrix α. Then α and A are both zero or $p_1 = p_2$, α and A are now singular, ω_1 and ω_2 are isomorphic.

For Ω_1 and Ω_2 are non-singular, $\bar{\alpha}\bar{\omega}_1 = \bar{\omega}_2 A$ and

$$\Omega_1 = \begin{pmatrix} \omega_1 \\ \bar{\omega}_1 \end{pmatrix}, \qquad \Omega_2 = \begin{pmatrix} \omega_2 \\ \bar{\omega}_2 \end{pmatrix}, \qquad \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix} = \Omega_1 A \Omega_2^{-1}. \tag{21}$$

If the rank of α is r then the rank of A is evidently $2r$. Moreover $r \leqq p_1$, $r \leqq p_2$. We let α and A be not both zero so that neither is zero and $r > 0$.

There exist non-singular matrices γ_1, γ_2, G_1, G_2 such that G_1 and G_2 are rational and

$$\alpha_0 = \gamma_2\alpha\gamma_1 = \begin{pmatrix} I(r) & 0 \\ 0 & 0 \end{pmatrix}, \qquad A_0 = G_2AG_1 = \begin{pmatrix} I(2r) & 0 \\ 0 & 0 \end{pmatrix}. \tag{22}$$

But then $\gamma_2\alpha\gamma_1\gamma_1^{-1}\omega_1G_1 = \gamma_2\omega_2G_2^{-1}G_2AG_1$. Let

$$\omega_{10} = \gamma_1^{-1}\omega_1G_1 = \begin{pmatrix} \omega_5 & \omega_6 \\ \omega_7 & \omega_8 \end{pmatrix}, \qquad \omega_{20} = \gamma_2\omega_2G_2^{-1} = \begin{pmatrix} \tau_5 & \tau_6 \\ \tau_7 & \tau_8 \end{pmatrix}, \tag{23}$$

where ω_5 and τ_5 have r-rows and $2r$-columns. Then $\alpha_0\omega_{10} = \omega_{20}A_0$ and

$$\begin{pmatrix} I(r) & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \omega_5 & \omega_6 \\ \omega_7 & \omega_8 \end{pmatrix} = \begin{pmatrix} \omega_5 & \omega_6 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \tau_5 & \tau_6 \\ \tau_7 & \tau_8 \end{pmatrix}\begin{pmatrix} I(2r) & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \tau_5 & 0 \\ \tau_7 & 0 \end{pmatrix}. \tag{24}$$

Then $\tau_5 = \omega_5$ and $\omega_6 = 0$, $\tau_7 = 0$. If $r < p_1$ then ω_1 is isomorphic to

$$\begin{pmatrix} \omega_5 & \omega_6 \\ 0 & \omega_8 \end{pmatrix}, \tag{25}$$

and is impure by Theorem 2. If $r < p_2$, then ω_2 is isomorphic to

$$\begin{pmatrix} \tau_5 & 0 \\ \tau_7 & \tau_8 \end{pmatrix}, \tag{26}$$

and is impure. Hence $r = p_1 = p_2$ so that α and A are non-singular and ω_1 and ω_2 are isomorphic to τ_5 and hence to each other.

Suppose that a matrix ω is isomorphic to ω_0 of (19), (20) and also to

$$\omega_{01} = \begin{bmatrix} M_1 & & 0 \\ & \ddots & \\ 0 & & M_s \end{bmatrix}, \qquad M_j = \begin{bmatrix} \tau_{j1} & & 0 \\ & \ddots & \\ 0 & & \tau_{js_j} \end{bmatrix}, \tag{27}$$

with $\tau_{jk} = \tau_{j1}$ and τ_{j1} not isomorphic to τ_{k1} for $j \neq k$. Let τ_{j1} be pure ($j = 1, \cdots, q$). If any τ_{j1} is isomorphic to an ω_{k1} then we may take $j = k$, $\tau_{j1} = \omega_{j1}$ without loss of generality by replacing ω_{01} by an isomorphic matrix. Hence we may assume that ω_{j1} and τ_{k1} are non-isomorphic for $j \neq k$ and are either non-isomorphic or equal for $j = k$. With this canonization we have

THEOREM 4. *The canonical form* (19) (20) *is unique.*

For ω_0 and ω_{01} are isomorphic and $\alpha\omega_0 = \omega_{01}A$ where α and A are non-singular. Then

$$(\alpha_{jk})\begin{bmatrix} \ddots & & \\ & \Lambda_k & \\ & & \ddots \end{bmatrix} = \begin{bmatrix} \ddots & & \\ & M_j & \\ & & \ddots \end{bmatrix}(A_{jk}) \quad (j = 1, \cdots, g; k = 1, \cdots, f), \tag{28}$$

so that $\alpha_{jk}\Lambda_k = M_j A_{jk}$. Similarly

$$(29) \qquad (\alpha_{j\,k}^{\lambda,\mu}) \begin{bmatrix} \ddots & & \\ & \omega_{k1} & \\ & & \ddots \end{bmatrix} = \begin{bmatrix} \ddots & & \\ & \tau_{j1} & \\ & & \ddots \end{bmatrix} (A_{j\,k}^{\lambda,\mu})$$

so that $\alpha_{j\,k}^{\lambda,\mu}\omega_{k1} = \tau_{j1}A_{j\,k}^{\lambda,\mu}$. Then $\alpha_{j\,k}^{\lambda,\mu}$ and $A_{j\,k}^{\lambda,\mu}$ are zero for $j \neq k$. If all $\alpha_{j\,j}^{\lambda,\mu}$ are zero, then α is singular. Hence $\alpha_{j\,j}^{\lambda,\mu} \neq 0$ for some λ, μ and by Theorem 3 and our canonization, we have $\omega_{j1} = \tau_{j1}$ and $f = g$. The matrix ω_{j1} has p_j rows and α_{jj} has $s_j p_j$ rows and $t_j p_j$ columns. If $s_j \neq t_j$, then

$$\alpha = \begin{bmatrix} \alpha_{11} & & 0 \\ & \ddots & \\ 0 & & \alpha_{ff} \end{bmatrix}$$

cannot be non-singular. For the determinant of α would be the product of a principal minor of α in α_{jj} and a matrix with either a row or column of zero elements. Hence $s_j = t_j$ and we have proved Theorem 4.

4. The multiplication algebra of a Riemann matrix is the algebra D of all multiplications

a: $$\alpha\omega = \omega A$$

of ω, where α is p-rowed and complex, A is $2p$-rowed and rational. If ω is a pure Riemann matrix then D is a division algebra by Theorem 3. Let then ω be impure of the form (19), (20), and let D_j be the multiplication algebra of ω_{j1}, a division algebra. Then we have

THEOREM 5. *The multiplication algebra of ω is a semi-simple algebra which is the direct sum of the multiplication algebras S_j of Λ_j. The algebras S_j are simple algebras $N_j \times D_j$ where D_j is the multiplication algebra of ω_{j1} and N_j is a total matric algebra of degree t_j.*

For let $\alpha\omega = \omega A$. As in (28) we have $\alpha_{jk} = 0$, $A_{jk} = 0$ $(j \neq k)$, $(j \neq k)$ so that D is evidently the direct sum of the algebras S_j of all multiplications $\alpha_{jj}\Lambda_j = \Lambda_j A_{jj}$ of Λ_j. Moreover D is semi-simple if the S_j are all simple. It is then sufficient to prove that the multiplication algebra of Λ_j is $D_j \times N_j$ since every such algebra is simple.

If $\alpha_{jj}\Lambda_j = \Lambda_j A_{jj}$ then $\alpha_{j\,j}^{\lambda,\mu}\omega_{j1} = \omega_{j1}A_{j\,j}^{\lambda,\mu}$ is a quantity of D_j and the multiplications of Λ_j are contained in the algebra $N_j \times D_j$ of all t_j-rowed matrices with elements in D_j. Conversely, every quantity a of $N_j \times D_j$ defines matrices $\alpha_{j\,j}^{\lambda,\mu}$, $A_{j\,j}^{\lambda,\mu}$ such that $\alpha_{j\,j}^{\lambda,\mu}\omega_{j1} = \omega_{j1}A_{j\,j}^{\lambda,\mu}$ is a quantity of D_j and if we define

$$\alpha_{jj} = (\alpha_{j\,j}^{\lambda,\mu}), \qquad A_{jj} = (A_{j\,j}^{\lambda,\mu}) \qquad (\lambda, \mu = 1, \cdots, t),$$

then the equation $\alpha_{jj}\Lambda_j = \Lambda_j A_{jj}$ states that a is a multiplication of Λ_j. Hence $S_j = N_j \times D_j$ as desired.

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ON THE CURVATURES OF CERTAIN CURVES IN FUNCTION SPACE[1]

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1. **Introduction.** The analogues in function space of the ordinary normals and curvatures of curves in euclidean space are well known.[2] Specifically, the general expressions for the curves of constant curvatures in euclidean n-space have been written down,[3] and the analogous formulas are of course valid for curves in function space with a finite number of constant curvatures different from zero. Definite results, however, relating to curves having an infinite sequence of curvatures, are lacking and it is the object of this paper to study some special curves of this nature.

It has seemed desirable to include a treatment of the differential equation for the curves with constant curvatures in n-dimensional space. The solution for this case is then generalized and it is shown that the class of curves in function space represented by the generalization really has an infinite sequence of curvatures all constant. The existence of curves of this nature is thus established by means of examples. The restrictions on the class of curves studied have been chosen with a view to securing convergence of the series involved and are probably more stringent than necessary.

The results just cited are very natural extensions of the corresponding situation in space of a finite number of dimensions. In one respect, however, the theory of curves with an infinite sequence of curvatures differs very materially from the finite case. It is no longer true, in general, that the sequence of curvatures essentially determines the curve. Nevertheless, many other properties are retained in the more general theory. We mention in particular the invariance of arc-length and of the sequence of curvatures under transformation of the parameter of the curve and also under transformation of the fundamental functions themselves, which are the extensions of translation and rotation in ordinary space.

2. **Definitions and Assumptions.** Corresponding somewhat to curves in ordinary space defined by a vector whose projections on the axes are of the form

$$y_i = f(i, u), \qquad (i = 1, 2, 3)$$

[1] The author wishes to acknowledge the kindly suggestions of Dr. Louis Ingold, Professor of Mathematics at the University of Missouri in the writing of this paper.

[2] Kowalewski, *Les formules de Frenet dans l'espace fonctionnel*, Comptes Rendus, vol. 151 (1910), p. 1338.

Ingold, L., *Functional Differential Geometry*, Transactions of the American Mathematical Society, vol. 13 (1912), p. 321.

[3] Brunel, *Sur les propriétés métriques des courbes gauches dans un espace linéaire à n dimensions*, Mathematische Annalen, vol. 19 (1882), p. 57.

we consider curves in a function space defined by a function,

$$\bar{f}(x, u) \qquad (a \leqq x \leqq b),$$

where x is a continuous parameter over the interval. Then we have a curve in space of infinitely many dimensions. We assume that the functions defining the curve have partial derivatives of all orders with respect to u, and that these together with the functions, themselves, are continuous functions of x over the interval.

In the general discussions of this paper it will be supposed that $\bar{f}(x, u)$ is expressible in the form

$$\bar{f}(x, u) = \sum_{j=1}^{\infty} \bar{A}_j(u)\, B_j(x)$$

where the $B_j(x)$ are a normed orthogonal system of functions, and that

$$\frac{\partial^n \bar{f}}{\partial u^n} = \sum_{j=1}^{\infty} \bar{A}_j^{[n]}(u)\, B_j(x)$$

and further that all the series involved are 'normally convergent.'[4] The convergence for the variable x is, of course, over the interval $(a \leqq x \leqq b)$. For the variable u the convergence relates to some finite interval. Set

$$\frac{ds}{du} = \int_a^b \left[\frac{\partial \bar{f}(x, u)}{\partial u}\right]^2 dx = \sum_{j=1}^{\infty} [\bar{A}_j'(u)]^2 \qquad \text{(See footnote 5)}$$

from which

$$(1) \qquad s = \int_{u_0}^{u} \sqrt{\sum_{i=1}^{\infty} [\bar{A}_i'(u)]^2}\, du\,.$$

If u is replaced by a function of s, $u = u(s)$, satisfying the last equation, $\bar{f}(x, u)$ becomes some function,

$$(2) \qquad f(x, s) = \sum_{i=1}^{\infty} A_i(s)\, B_i(x)$$

such that

$$(3) \qquad \int_a^b \left[\frac{\partial f(x, s)}{\partial s}\right]^2 dx = 1\,. \qquad \text{(See footnote 6)}$$

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[4] Uniformly convergent according to the M-test. Bromwich, *Theory of Infinite Series*, p. 124.

[5] The integrated product of two functions corresponds to the ordinary inner product of two vectors.

[6] Ingold, loc. cit., p. 320.

Then s is said to be arc-length along the curve defined by $f(x, s)$. When $f(x, s)$ is a series of the form (2) it is necessary that

$$\sum_{i=1}^{\infty} A_i'(s)^2 = 1 .$$

The derivative, $\dfrac{\partial f(x, s)}{\partial s}$, is called a *tangent to the curve f*. Corresponding to the set of normals at a point of a curve in ordinary space, the tangent, principal normal, and binormal, we set up a system of expressions defining the "normals" to f. They are extensions of the well-known Frenet formulas,

$$n_0(x, s) = t(x, s) = \frac{\partial f(x, s)}{\partial s} ,$$

$$k_i(s)\, n_i(x, s) = \frac{\partial n_{i-1}(x, s)}{\partial s} + k_{i-1}(s)\, n_{i-2}(x, s) . \tag{5}$$

$$(i \geqq 1) , \qquad n_{-1}(x, s) \equiv 0 , \qquad k_0(s) = 1 .$$

The $k_i(s)$ are called the curvatures, and their reciprocals $1/k_i(s)$, the radii of curvature. The curvatures are defined by the equations,

$$k_i^2(s) = \int_a^b \left[\frac{\partial n_{i-1}(x, s)}{\partial s} + k_{i-1}(s)\, n_{i-2}(x, s) \right]^2 dx . \qquad (i \geqq 1) . \tag{6}$$

It has been proved that the above system of so-called 'normals' is mutually orthogonal, that

$$\int_a^b n_i^2(x, s)\, dx = 1 ,$$

and that when one curvature vanishes, all the succeeding ones also vanish.[7]

3. **The Curves of Constant Curvatures in an n-Space Imbedded in Function Space.** Given a set of any $n - 1$ positive constant curvatures none of which are infinite or zero, it is possible to write down the differential equation of the curves in n-space possessing those curvatures. By recursion any one normal can be expressed linearly in terms of the vector, f, and its derivatives with respect to s. It is then readily proved by induction that the differential equation of the curves of constant curvatures, $k_1, k_2, \cdots, k_{n-1}$, is, in the differential operator notation,

$$\left[D^{n+1} + \sum_{i=1}^{[n/2]} \left(\sum_{\alpha} k_{\alpha_1}^2\, k_{\alpha_2}^2 \cdots k_{\alpha_i}^2 \right) D^{n+1-2i} \right] f = 0 . \tag{7}$$

$$\alpha_i \geqq 2 + \alpha_r \qquad \text{for} \qquad i > r. \qquad \text{Also} \qquad D^0 = 1 .$$

[7] Ingold, loc. cit., pp. 321-3.

From the form of this differential equation we see the solution to be

$$f(x, s) = \sum_{i=1}^{[n/2]} \left[\frac{m_i}{h_i} \sin (h_i s) \cdot B_{2i-1}(x) + \frac{m_i}{h_i} \cos (h_i s) \cdot B_{2i}(x) \right] + [1 - (-1)^n] \frac{cs}{2} \cdot B_n(x) . \quad (8)$$

[See Footnote 8]

The $\frac{m_i}{h_i} B_j(x)$, being clearly independent of s, replace the usual arbitrary constants in the solution and have the form given in order to give the restrictions on them the conciseness as of the next section. The signs are all taken to be positive as the curve is essentially unchanged by so doing, and the constant of integration is omitted as it represents only a translation of the curve.

If n is odd, then clearly $cs \cdot B_n(x)$ is a particular solution of the differential equation. Hence we add $cs \cdot B_n(x)$ to the vector form of $f(x, s)$. In an odd space the curve has the property of a circular helix of extending to infinity along one axis. In even space the curve is contained entirely within a hyper-rectangular-parallelopiped and may or may not be closed. The h_i are constants obtained from the auxiliary equation to the differential equation. In the expressions for the curvatures we have the equations for determining the m_i.

Since the unit normals may be directed either positively or negatively the Frenet formulas do not fix the signs of the normals and curvatures. Let us assume that any changes of sign of the set $k_1, \cdots, k_i$ do not numerically affect others of the set. We shall show that the set may be extended to include k_{i+1}. Suppose that after certain changes k_i takes the opposite sign. Now

$$n_i k_i = n'_{i-1} + n_{i-2} k_{i-1} , \quad (9)$$

$$n_{i+1} k_{i+1} = n'_i + n_{i-1} k_i . \quad (10)$$

If n_i is unchanged, n'_{i-1} must be changed and n_{i-2} or k_{i-1} changed (from (9)). Then n_{i-1} has its sign changed, and $n_{i-1} k_i$ is unchanged. And since n_i is unchanged n'_i is unchanged and the numerical value of k_{i+1} is unchanged. On the other hand, if when k_i takes the opposite sign n_i changes sign, then n'_{i-1} is unchanged and therefore also n_{i-1}. But n'_i and k_i both change sign so again the numerical value of k_{i+1} is unchanged. Since the assumption is easily proved for k_1 and k_2, the induction is complete.

After applying the definitions (6) of the curvatures to our curve of constant curvatures, we obtain

$$k_1^2 = \sum_{i=1}^{[n/2]} m_i^2 h_i^2 ,$$

$$k_2^2 = \frac{1}{k_1^2} \sum_{i=1}^{[n/2]} m_i^2 [k_1^2 - h_i^2]^2 + \frac{[1 - (-1)^n]}{2} \left(1 - \sum_{j=1}^{[n/2]} m_j^2 \right) k_1^2 . \quad (11)$$

[8] Brunel, loc. cit.

The general formula proved by induction is

$$k_t^2 = \prod_{j=1}^{t-1} \frac{1}{k_j^2}$$

$$(12) \qquad \sum_{i=1}^{[n/2]} \left[m_i \, h_i{}^{\delta^{t-1}_{2[t/2]}} \sum_{r=0}^{[t/2]} (-1)^r \, h_i^{2r} \sum_{\alpha} (k_{\alpha_1} \, k_{\alpha_2} \cdots k_{\alpha_{[t/2]-r}})^2 \right]^2 .$$

$$k_{\alpha_0} = 1 . \qquad \alpha_i \geqq 2 + \alpha_j \qquad \text{for} \qquad i > j . \qquad t > 2 .$$

4. **Curves With an Infinite Sequence of Non-Vanishing Curvatures.** The formulas just given for the function $f(x, s)$ and for the curvatures hold when n is any finite positive integer. It seems natural to attempt to extend these formulas to the case in which there exists an infinite sequence of non-vanishing curvatures. In this case we write in place of formula (8)

$$(13) \qquad f(x, s) = cs \cdot B_1(x) + \sum_{i=1}^{\infty} \frac{m_i}{h_i} [\sin (h_i s) \cdot B_{2i}(x) + \cos (h_i s) \cdot B_{2i+1}(x)]$$

where

$$c^2 + \sum_{i=1}^{\infty} m_i^2 = 1 .$$

If $c \neq 0$ we say the space of the curve is odd, if $c = 0$ it is even. In the example to be used $c = 0$. We write in place of (12), with similar changes in (11) obtained directly from (6),

$$k_t^2 = \prod_{j=1}^{t-1} \frac{1}{k_j^2}$$

$$(14) \qquad \sum_{i=1}^{\infty} \left[m_i \, h_i{}^{\delta^{t-1}_{2[t/2]}} \sum_{r=0}^{[t/2]} (-1)^r \, h_i^{2r} \sum_{\alpha} (k_{\alpha_1} \, k_{\alpha_2} \cdots k_{\alpha_{[t/2]-r}})^2 \right]^2 .$$

$$k_{\alpha_0} = 1 . \qquad \alpha_i \geqq 2 + \alpha_j \qquad \text{for} \qquad i > j . \qquad t > 2 .$$

This formula may be written more compactly by introducing the following notations:

$$k_0 = 1 , \qquad A_0^i = 1 , \qquad A_1^i = 1 ,$$

$$A_r^i = \frac{k_{r-1}}{k_{r-2}} A_{r-2}^i + (-1)^{r+1} \frac{A_{r-1}^i}{k_{r-1}} h_i^{2 \delta^r_{2[r/2]}} , \qquad (r = 2, 3, \cdots) .$$

Then (14) becomes

$$(15) \qquad k_r^2 = \sum_{i=1}^{\infty} (m_i \, A_r^i \, h_i^{\delta^{r-1}_{2[r/2]}})^2 , \qquad (r = 1, 2, \cdots) .$$

It is clearly possible to choose the constants m_i and h_i so that the series (13) converges. Then it will define a function of x and s and hence a curve. In order that the parameter s shall be arc-length of the curve defined by (13) it is sufficient that the series obtained by differentiation of (13) with respect to s converge 'normally' with respect to s over some given interval and that $\sum_{i=1}^{\infty} m_i^2 = 1$. The following is such an example:

$$f(x, s) = \sum_{i=1}^{\infty} \frac{a}{10^i}\left[\sin \frac{s}{10^i} \cdot \cos (2i - 1)\, x + \cos \frac{s}{10^i} \cdot \cos 2ix\right],$$

where

$$a = \sqrt{\frac{19{,}998}{\pi}}.$$

This has the form of (13) where $m_i^2 = 9{,}999(\cdot 1)^{4i}$ and $h_i = (\cdot 1)^i$. $B_1(x)$, $B_2(x)$, $\cdots$ is the system of functions, $\sqrt{\frac{2}{\pi}} \cos x$, $\sqrt{\frac{2}{\pi}} \cos 2x$, $\cdots$, normed and orthogonal over the interval $(0 \leqq x \leqq \pi)$. The above example can be proved to be 'normally' convergent in x and s over their respective intervals. It is then possible to show that each of the curvatures of the infinite sequence of curvatures exists and is different from zero. Since $k_{j+1}\, n_{j+1}$ is defined in terms of the previous normals and curvatures, the existence of n_{j+1} will be established when we have shown that k_{j+1} exists. This we proceed to do.

Assume that all the series representing $n_0(x, s)$, $n_1(x, s)$, $\cdots$, $n_j(x, s)$, $\frac{\partial^{[j-r+1]}\, n_r(x, s)}{\partial s^{[j-r+1]}}$ for $r = 0, 1, \cdots, j$, and $k_1^2, k_2^2, \cdots, k_j^2$ are 'normally' convergent and that $k_1^2, k_2^2, \cdots, k_j^2 \neq 0$. Now

$$k_{j+1}^2 = \int_0^{\pi} [k_{j+1}\, n_{j+1}]^2\, dx = \int_0^{\pi}\left[\frac{\partial n_j}{\partial s} + k_j\, n_{j-1}\right]^2 dx \text{ if it exists.}$$

Since by the assumptions the latter integral exists, k_{j+1} exists, and the series for $k_{j+1}\, n_{j+1}$ converges 'normally.'

n_j involves alternate sin $(h_i s)$ and cos $(h_i s)$ terms. $\frac{\partial n_{j+1}}{\partial s}$ involves only one part, $\frac{\partial}{\partial s}\left[\frac{\partial n_j}{\partial s}\right]$, which is not assumed 'normally' convergent. But the absolute value of each term of the series, $\frac{\partial}{\partial s}\left[\frac{\partial n_j}{\partial s}\right]$, is clearly less than that for the corresponding term in the series for n_j since $h_i < 1$. Hence this part and consequently $\frac{\partial n_{j+1}}{\partial s}$ converge 'normally.'

It remains to be shown that no curvature of finite order vanishes. If k_{j+1}^2 vanishes when $j + 1$ is finite, then each parenthesis expression[9] in the formula

[9] See formulas (14) and (15).

for k_{j+1}^2 must vanish for an infinite number of different values of h_i^2. Since the expression is an algebraic one of finite degree in h_i^2 that is impossible. Therefore no finite curvature vanishes. The induction is complete when we observe that for $j = 2$, n_0, n_1, n_2, and their required derivatives with respect to s, k_1^2 and k_2^2 converge 'normally.'

This discussion relates to functions of the type (13) with $c = 0$ which are the extensions, to an infinite number of dimensions, of the formulas for curves of constant curvatures in finite space of an even number of dimensions. It is clearly possible to carry through a similar discussion for curves of type (13) with $c \neq 0$, and hence both types of curves exist in the space of the set of normed orthogonal functions, $B_i(x)$. The two cases may be called the even case and the odd case.

It is interesting to note that for the even case if the h_i are all the same constant, the second curvature vanishes and consequently the curve is a plane curve. Under the same conditions for the odd case the third curvature vanishes and the curve lies in a three-dimensional space. This can be shown by direct substitution in the extensions of (11).

5. **Some Invariant Properties of a Curve in Function Space.** For curves in a finite number of dimensions it is known that the curvatures and length of arc are invariant under a certain class of transformations of the containing space (as translations, rotations, etc.), and also under analytic transformations of the parameter of the curve. We give in conclusion a brief treatment of the corresponding ideas for curves in function space having an infinite sequence of curvatures.

Let $f(x, s) = \sum_{i=1}^{\infty} A_i(s)\ B_i(x)$ and $\sum_{i=1}^{\infty} [A_i'(s)]^2 = 1$ so that s is arc-length. Suppose the curve, $f(x, s)$, has an infinite sequence of curvatures and normals which can be obtained from the Frenet formulas by using term-wise differentiation of the series on the right, and that the resulting series as well as the series for $f(x, s)$ are 'normally' convergent in both variables. Let the $B_i(x)$ form a closed set, normed and orthogonal over a certain interval.

If $\theta_i(x)$ form an equivalent closed set[10] of normed orthogonal functions (the B's themselves, reordered, are a special set of this type) and if the same conditions hold on the series $\bar{f}(x, s) = \sum_{i=1}^{\infty} A_i(s)\ \theta_i(x)$ and the series obtained from it as on $f(x, s) = \sum_{i=1}^{\infty} A_i(s) B_i(x)$, it is true that $\bar{f}(x, s)$ has the same infinite sequence of curvatures as $f(x, s)$. This is so, since if they exist, the curvatures depend only on the ordered sequence of the A's.

[10] We call two infinite closed sets equivalent if the elements of either are expressible linearly in terms of the elements of the other.

Let us subject the B's to the transformation

$$B_l(x) = \sum_{j=1}^{\infty} b_{j\,l}\,\theta_j(x) \tag{16}$$

where

$$\sum_{k=1}^{\infty} b_{k\,l}^2 = 1 \qquad \text{and} \qquad \sum_{k=1}^{\infty} b_{k\,l}\,b_{k\,m} = 0\,, \qquad l \neq m\,. \tag{17}$$

$f(x, s)$ then becomes some function

$$\bar{f}(x, s) = \sum_{i=1}^{\infty} A_i\left[\sum_{j=1}^{\infty} b_{j\,i}\,\theta_j\right] = \sum_{j=1}^{\infty}\left[\sum_{i=1}^{\infty} b_{j\,i}\,A_i\right]\theta_j = \sum_{j=1}^{\infty} \bar{A}_j(s)\,\theta_j(x)\,,$$

(See footnote 11)

where

$$\bar{A}_j = \sum_{i=1}^{\infty} b_{j\,i}\,A_i\,. \tag{18}$$

The questions arise as to whether or not s is still arc-length and whether or not $\bar{A}_j(s)$ is equivalent to the old $A_j(s)$. In other words, does $\bar{f}(x, s)$ have the same infinite sequence of curvatures as $f(x, s)$?

Assume that derivatives of all orders of $\bar{A}_j$ with respect to s exist. If $\sum_{j=1}^{\infty}\left[\frac{\partial \bar{A}_j}{\partial s}\,\theta_j\right]$ is 'normally' convergent and consequently equal to $\frac{\partial}{\partial s}\left[\sum_{j=1}^{\infty} \bar{A}_j\,\theta_j\right] = \frac{\partial \bar{f}(x, s)}{\partial s}$, then $\sum_{j=1}^{\infty} [\bar{A}_j'(s)]^2 = 1$ and s is still arc-length. For

$$\sum_{j=1}^{\infty} [\bar{A}_j'(s)]^2 = \sum_{j=1}^{\infty}\left[\sum_{i=1}^{\infty} b_{j\,i}\,A_i'\right]^2$$

$$= \sum_{j=1}^{\infty} [b_{j\,1}\,A_1' + b_{j\,2}\,A_2' + \cdots]^2 = A_1'^2 + A_2'^2 + \cdots$$

by (17), and this equals one by hypothesis.

In order for the sequence of normals to exist in the new form it is necessary to suppose that $\frac{\partial^n \bar{f}}{\partial s^n} = \sum_{j=1}^{\infty} \bar{A}_j^{[n]}(s)\,\theta_j(x)$, and further that the series, $\sum_{k,l} \bar{A}_k^{[r]}\bar{A}_l^{[t]}\theta_k\theta_l$, converge 'normally' in both variables; then $\bar{k}_i\,\bar{n}_i$ can be computed from the series for $\bar{f}(x, s)$.

If

$$k_r\,n_r = \sum_{i=1}^{\infty} [a_{r+1}\,A_i^{[r+1]} + \cdots + a_1\,A_i']\,B_i$$

and

$$\bar{k}_r\,\bar{n}_r = \sum_{i=1}^{\infty} [a_{r+1}\,\bar{A}_i^{[r+1]} + \cdots + a_1\,\bar{A}_i']\,\theta_i\,,$$

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[11] $\sum_{i=1}^{\infty} A_i\left[\sum_{j=1}^{\infty} b_{ji}\,\theta_j\right]$ must be summable by rows or columns.

where the a_t are functions of s, we shall say they are 'similar.' Assume that $k_i n_i$ and $\bar{k}_i \bar{n}_i$ are 'similar' and $k_i^2 = \bar{k}_i^2$ for $i = 1, 2, \cdots, j$. Since $k_{j+1} n_{j+1} = \frac{\partial n_j}{\partial s} + k_j n_{j-1}$ it is easily seen that $k_{j+1} n_{j+1}$ is 'similar' to $\bar{k}_{j+1} \bar{n}_{j+1}$. We now show that $\bar{k}_{j+1}^2 = k_{j+1}^2$.

In obtaining the curvatures the B's and θ's integrate out giving

$$k_{j+1}^2 = \sum_{i=1}^{\infty} [a_{j+2} A_i^{[j+2]} + a_{j+1} A_i^{[j+1]} + \cdots + a_1 A_i']^2 , \tag{19a}$$

$$\bar{k}_{j+1}^2 = \sum_{i=1}^{\infty} [a_{j+2} \bar{A}_i^{[j+2]} + a_{j+1} \bar{A}_i^{[j+1]} + \cdots + a_1 \bar{A}_i']^2 . \tag{19b}$$

Make substitutions of the type $\bar{A}_i^{[r]} = \sum_{l=1}^{\infty} b_{il} A_l^{[r]}$ throughout (19b). Then let us sum as to i (as indicated) the general term of the squared parenthesis of (19a),

$$\sum_{i=1}^{\infty} a_{\mu_1} a_{\mu_2} A_i^{[\mu_1]} A_i^{[\mu_2]}$$

and form the like sum for the corresponding term in (19b), or

$$\sum_{i=1}^{\infty} a_{\mu_1} a_{\mu_2} \bar{A}_i^{[\mu_1]} \bar{A}_i^{[\mu_2]} = \sum_{i=1}^{\infty} a_{\mu_1} a_{\mu_2} \left[\sum_{l=1}^{\infty} b_{il} A_l^{[\mu_1]} \right] \left[\sum_{t=1}^{\infty} b_{it} A_t^{[\mu_2]} \right]$$

$$= \sum_{i=1}^{\infty} a_{\mu_1} a_{\mu_2} [b_{i1} A_1^{[\mu_1]} + b_{i2} A_2^{[\mu_1]} + \cdots] [b_{i1} A_1^{[\mu_2]} + b_{i2} A_2^{[\mu_2]} + \cdots] .$$

Because of (17) the summation on any cross-product term of the two latter brackets vanishes. The entire expression therefore reduces to

$$\sum_{i=1}^{\infty} a_{\mu_1} a_{\mu_2} A_i^{[\mu_1]} A_i^{[\mu_2]},$$

which is the general term in the expanded form of $\bar{k}_{j+1}^2$ and also of k_{j+1}^2. Therefore $\bar{k}_{j+1}^2$ equals k_{j+1}^2. Since the assumptions are readily shown to be true for $j = 1$ and $j = 2$ the induction is complete.

Thus, if a transformation of the type (16) is made on $f(x, s) = \sum_{i=1}^{\infty} A_i(s) B_i(x)$, then s remains arc-length and the two curves have the same infinite sequence of curvatures, provided they exist.

It is readily seen that if the curve

$$f(x, s) = \sum_{i=1}^{\infty} A_i(s) B_i(x)$$

has an infinite sequence of curvatures obtained from formula (2) by using the successive derivatives of the series on the right, then the curve,

$$\bar{f}(x, s) = \sum_{i=1}^{\infty} A_i(s) \theta_i(x)$$

has exactly the same sequence of curvatures, (provided its curvatures may be computed from the series on the right), *where the functions* $\theta_i(x)$ *are any infinite sub-set of the fundamental set* $B_i(x)$, because the curvatures depend only on the ordered sequence of coefficients $A_i(s)$. It follows that if there exists a curve in the space of the functions, $B_1(x), B_2(x), \cdots$ having a given infinite sequence of curvatures there are infinitely many such curves. As an immediate consequence it is clear that curves in function space are not essentially determined uniquely by their sequence of curvatures as they are in space of a finite number of dimensions, but there is a whole class of curves determined from them by the methods indicated above.

Furthermore, by a proof identical to that given in this section it can be shown that arc-length and the infinite sequence of curvatures, if they can be computed from the transformed series, are invariants of $\bar{f}(x, s)$ given above, under transformations on the $\theta_i(x)$ of the type (16), for every curve of the class just mentioned.

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REMARKS ON SOME POINTS IN THE THEORY OF DIVERGENT SERIES

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1. I have collected here a number of remarks which have occurred to me when lecturing on Fourier series or on the general theory of divergent series. They contain nothing that is important, and not a great deal that is new, but may be interesting because they refer to comparatively unfamiliar topics.

A. On Fourier's own deduction of Fourier's theorem

2.1. By 'Fourier's theorem' I mean here the theorem that, if $f(x)$ belongs to an appropriate class of functions, and is 'representable' by a trigonometric series

$$\tfrac{1}{2}a_0 + \sum_1^\infty (a_n \cos nx + b_n \sin nx), \tag{2.1.1}$$

then a_n and b_n are given by the formulae

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx. \tag{2.1.2}$$

The formulae are older than Fourier, but with that I am not concerned.[1]

'Fourier's theorem' asserts, in modern language, that a trigonometrical series which represents the function must be the 'Fourier series' of the function. It is true or false according to the class of functions considered and the manner in which the function is 'represented' by the series. Thus it is true, after du Bois-Reymond and de la Vallée-Poussin, when the function is bounded or integrable and the series is convergent. If we assume only that the series is summable, by one or other of the standard methods of the theory of divergent series, then the theorem is usually false, even when the function is 0.[2] Thus the series

$$\sin x + 2\sin 2x + 3\sin 3x + \cdots \tag{2.1.3}$$

is summable to sum 0 by the Abel-Poisson method or by Cesàro means of any order greater than 1.

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[1] The fullest account of the early history of the formulae is that given by Burkhardt, 'Trigonometrische Reihen und Integrale', *Encykl. d. Math. Wiss.*, II. A. 12 (in particular §16). Burkhardt traces the first of them back to Clairaut (1757). The usual deduction by term-by-term integration is due to Euler (1777).

[2] The best results in this direction are due to Rajchmann, Zygmund, and Verblunsky. In particular Verblunsky [*Proc. London Math. Soc.* (2), 31 (1930) and 34 (1932)] has shown that a trigonometrical series cannot be summable $(C, 1)$ to 0 unless its coefficients are nul, thus confirming a conjecture of M. Riesz [*Math. Ann.* 71 (1911)].

Here I am concerned primarily with the very special case which is considered by Fourier in §§207–218 of his *Théorie de la chaleur*. Suppose that $f(x)$ is odd and analytic for $|x| \leqq \pi$, so that

$$f(x) = \sum_0^\infty \frac{f^{(2h+1)}(0)}{(2h+1)!} x^{2h+1} \tag{2.1.4}$$

for $|x| \leqq \pi$, and that

$$f(x) = \sum_1^\infty b_n \sin nx \tag{2.1.5}$$

for $-\pi < x < \pi$, the series (2.1.5) being convergent in the classical sense.[3] These are in effect Fourier's assumptions, and his object is to prove that b_n is given by the second of formulae (2.1.2). It will be observed that he is attempting to prove a theorem of 'uniqueness'. He wishes to show not merely (what is trivial now) that $f(x)$ can be expanded in the form required, but also that the series so obtained is the *only* trigonometrical series for the function. The conclusion desired is true, but only on account of 'Cantor's theorem'; and a mild generalization of the interpretation of (2.1.5) would render the conclusion false.

Fourier replaces every sine in (2.1.5) by its Taylor series, and equates coefficients.[4] He thus obtains the system of equations

$$b_1 + 2^{2h+1}b_2 + 3^{2h+1}b_3 + \cdots = (-1)^h f^{(2h+1)}(0) \qquad (h = 0, 1, 2, \cdots). \tag{2.1.6}$$

It will be observed that the series here are all divergent even in the simplest cases; thus the Fourier series of $f(x) = x$ is

$$2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right),$$

and the system (2.1.6) then becomes

$$1 - 1 + 1 - \cdots = \tfrac{1}{2}, \qquad 1 - 2^{2h} + 3^{2h} - \cdots = 0 \qquad (h = 1, 2, \cdots).$$

These equations are correct when properly interpreted, but they require a theory of divergent series; and a very little use of divergent series is, as we have seen, sufficient to falsify Fourier's conclusion.

Fourier next replaces the system (2.1.6) by the corresponding system

$$\beta_1 + 2^{2h+1}\beta_2 + \cdots + k^{2h+1}\beta_k = (-1)^h f^{(2h+1)}(0) \qquad (h = 0, 1, \cdots, k-1) \tag{2.1.7}$$

of k rows and columns, solves this system for $\beta_1, \beta_2, \cdots, \beta_k$ and makes k tend to infinity. The method, as Riesz remarks, embodies an important principle,

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[3] The sum of the series at the ends of the interval will naturally not be $f(x)$ but

$$\tfrac{1}{2}\{f(-\pi) + f(\pi)\} = 0.$$

[4] There is a short but very interesting discussion of Fourier's argument in F. Riesz, *Les systèmes d'équations linéaires à une infinité d'inconnues* (1913), pp. 2–7.

and the calculation is the more remarkable because Fourier had no determinants to help him. It would be hopeless to attempt to 'justify' the method, in this particular application, under any 'reasonable' conditions, and Fourier's calculations are now only a historical curiosity. But the *result*, viz.

$$\frac{1}{2}\pi b_n = \frac{(-1)^{n-1}}{n}\left\{f(\pi) - \frac{f''(\pi)}{n^2} + \frac{f''''(\pi)}{n^4} - \cdots\right\}, \tag{2.1.8}$$

is true fairly generally, and is a formula of considerable interest.

2.2. The formulae (2.1.7) and (2.1.8) suggest several different questions. The first is whether (2.1.8) is in fact a valid representation of the Fourier coefficient of $f(x)$.

In the first place, if $f(x)$ is analytic for $-\pi \leqq x \leqq \pi$,[5] we have

$$\frac{1}{2}\pi b_n = \int_0^\pi f(x)\sin nx\,dx = (-1)^{n-1}\left\{\frac{f(\pi)}{n} - \frac{f''(\pi)}{n^3} + \cdots + (-1)^h\frac{f^{(2h)}(\pi)}{n^{2h+1}} + \frac{(-1)^h}{n^{2h+1}}\int_0^\pi f^{(2h+1)}(x)\cos nx\,dx\right\},$$

by repeated partial integration. Since the last term is $O(n^{-2h-2})$ for large n, we conclude that *the series* (2.1.8) *is an asymptotic series for* b_n, *for any analytic* $f(x)$.

The series will be convergent, for sufficiently large n, if

$$|f^{(2h)}(\pi)| < K^{2h}$$

for some K. For this, it is necessary and sufficient that $f(x)$ be an integral function of order 1 and finite type, *i.e.* that

$$|f(x)| < Ae^{B|x|} \tag{2.2.1}$$

for some A and B. If $B < 1$, in which case also $K < 1$, then the series is convergent for $n \geqq 1$. All this is familiar.[6]

We are led to ask whether the series

$$\frac{f(\pi)}{n} - \frac{f''(\pi)}{n^3} + \frac{f''''(\pi)}{n^5} - \cdots \tag{2.2.2}$$

or

$$\frac{f(\pi)}{n} + 0 - \frac{f''(\pi)}{n^3} + 0 + \frac{f''''(\pi)}{n^5} + \cdots \tag{2.2.3}$$

are summable by recognised methods: it is the second series which presents itself most naturally. If $f(x)$ is an integral function, we have

$$\frac{f(\pi)}{n} - \frac{f''(\pi)}{n^3}\frac{t^2}{2!} + \frac{f''''(\pi)}{n^5}\frac{t^4}{4!} - \cdots = \frac{1}{2n}\left\{f\left(\pi + \frac{it}{n}\right) + f\left(\pi - \frac{it}{n}\right)\right\}, \tag{2.2.4}$$

[5] That is to say, in a region of the complex plane containing the portion of the real axis from $-\pi$ to π. This is naturally a more general hypothesis than Fourier's.

[6] See in particular G. Pólya, *Math. Zeitschrift*, 29 (1929), 549–640 (especially pp. 578–580).

and so

$$(2.2.5)\qquad \int_0^\infty e^{-t}\left\{\frac{f(\pi)}{n}-\frac{f''(\pi)}{n^3}\frac{t^2}{2!}+\cdots\right\}dt$$
$$=\frac{1}{2n}\int_0^\infty e^{-t}\left\{f\left(\pi+\frac{it}{n}\right)+f\left(\pi-\frac{it}{n}\right)\right\}dt$$
$$=\frac{1}{2}\int_0^\infty e^{-nt}\{f(\pi+it)+f(\pi-it)\}\,dt,$$

if the integral is convergent. In these circumstances the series (2.2.3) is summable (B), *i.e.* by Borel's exponential integral. The series (2.2.2) is summable in the sense which I have called $(B, 2)$.[7] It is summable $(B, 1)$, *i.e.* (B), whenever

$$\frac{f(\pi)}{n}-\frac{f''(\pi)}{n^3}\frac{t}{1!}+\frac{f''''(\pi)}{n^5}\frac{t^2}{2!}-\cdots$$

is convergent for all t; for example, when $f(x)$ is of finite type, in which case the series is convergent for large n and summable (B) for all positive n.

Suppose now that $f(x)=f(\xi+i\eta)$ is (integral and) of finite type in the half-strip R defined by

$$-\pi\leqq\xi\leqq\pi,\ \eta\geqq 0;$$

that is to say that (2.2.1) is satisfied in R. Then, applying Cauchy's theorem to

$$\int e^{nix}f(x)\,dx$$

and R, we obtain

$$\pi b_n=\frac{1}{i}\int_{-\pi}^{\pi}e^{nix}f(x)\,dx$$
$$(2.2.6)\qquad =\int_0^\infty e^{-n\pi i-n\eta}f(-\pi+i\eta)\,d\eta-\int_0^\infty e^{n\pi i-n\eta}f(\pi+i\eta)\,d\eta$$
$$=(-1)^{n-1}\int_0^\infty e^{-n\eta}\{f(\pi+i\eta)+f(\pi-i\eta)\}\,d\eta.$$

Comparing (2.2.5) and (2.2.6), we see that the series (2.2.3) is summable (B) to sum $(-1)^{n-1}\frac{1}{2}\pi b_n$. That is to say, when $f(x)$ satisfies the conditions stated, *Fourier's formula is correct for sufficiently large n if the infinite series is interpreted as a Borel sum. It is correct in this sense for all n if the B of* (2.2.1) *is less than* 1. For example, these conditions are satisfied when

$$f(x)=x\cos ax^2 \qquad (2a\pi<1).$$

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[7] See G. H. Hardy, *Journal London Math. Soc.*, 8 (1934), 153–157. A series is summable (B, k) when the series obtained by interpolating $k-1$ zeros between each pair of successive terms is summable (B).

It is to be observed that the series may be summable, or even converge, without representing b_n; thus it vanishes identically whenever $f(x)$ has the period 2π. In this case $f(x)$, if of finite type, is a trigonometrical polynomial, and $b_n = 0$ for sufficiently large n.

There is another generalization of Borel's method which is fairly familiar and which enables us to extend the range of validity of Fourier's formula further. The generalization is that often used in summing the series

$$\sum a_n x^n = 1 - 1!\, x + 2!\, x^2 - \cdots .$$

In this case the series

$$\sum a_n \frac{(xt)^n}{n!} = 1 - xt + x^2t^2 - \cdots$$

is convergent only when $t \mid x \mid < 1$, but is summable (B) when $\Re(xt) \geqq -1$, *i.e.* when xt lies within the Borel polygon of summability. This condition will be satisfied, for all positive t, if and only if $\Re(x) > 0$; and then we may write

$$1 - 1!\, x + 2!\, x^2 - \cdots = \int_0^\infty \frac{e^{-t}}{1 + xt}\, dt ,$$

and say that the series is summable (B^2), *i.e.* by a repeated application of Borel's method.

We assumed that $f(x)$ is an integral function only in writing down the sum of the series in (2.2.4). This equation will be true, in the Borel sense, whenever all the arguments of $f(x)$ in question lie within the Borel polygon of the series; and this will be true, for every t, if *all the singularities of $f(x)$ are on the real axis and at a distance greater than π from the origin.* If this condition is fulfilled, and $f(x)$ satisfies (2.2.1) in R, then our conclusions about Fourier's formula remain valid with the substitution of (B^2) for (B).

As an example, suppose

$$f(x) = \frac{1}{a - x} - \frac{1}{a + x} \qquad (a > \pi) .$$

2.3. There remain Fourier's formulae (2.1.6). Fourier's series are divergent in the most trivial cases, but the simplest definitions are sufficient to sum them and to re-establish the formulae. Riesz,[8] for example, remarks that the formulae are correct if the series are interpreted as Abel-Poisson sums. Much more indeed is true, since Young[9] has shown that the k^{th} derived series of the Fourier series of $f(x)$ is summable (C, k), at any internal point, whenever $f^{(k)}(x)$ is continuous. However, the Abel-Poisson method is the simplest and most natural which will sum *all* the series, and it may be worth while to give a very simple proof that it will do so.

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[8] l. c., p. 5.

[9] *Proc. London Math. Soc.* (2) 17 (1918), 195–236.

Since $f(x)$ is regular for $-\pi \leqq x \leqq \pi$, we have

$$b_n = \frac{1}{\pi i}\int_{-\pi}^{\pi} f(x)\, e^{nix}\, dx = \frac{1}{\pi i}\int_{C_1} f(x)\, e^{nix}\, dx,$$

where C_1 is a curve from $-\pi$ to π a little above the real axis. Hence, if $\delta > 0$,

$$\sum b_n e^{-\delta n} = \frac{1}{\pi i}\int_{C_1} f(x) \sum e^{nix-\delta n}\, dx = \frac{1}{\pi i}\int_{C_1} f(x)\, \frac{e^{ix-\delta}}{1-e^{ix-\delta}}\, dx.$$

Differentiating $2h+1$ times with respect to δ, and then replacing the derivative under the integral sign by the corresponding derivative with respect to x, we obtain

$$\sum n^{2h+1} b_n e^{-\delta n} = \frac{(-1)^{h-1}}{\pi}\int_{C_1} f(x)\left(\frac{d}{dx}\right)^{2h+1} \frac{e^{ix-\delta}}{1-e^{ix-\delta}}\, dx.$$

When $\delta \to 0$, the right hand side tends to

$$\frac{(-1)^{h-1}}{\pi}\int_{C_1} f(x)\left(\frac{d}{dx}\right)^{2h+1} \frac{e^{ix}}{1-e^{ix}}\, dx = \frac{(-1)^h}{2\pi i}\int_{C_1} f(x)\left(\frac{d}{dx}\right)^{2h+1} \cot\frac{1}{2}x\, dx$$

$$= \frac{(-1)^h}{2\pi i}\int_{C} f(x)\left(\frac{d}{dx}\right)^{2h+1} \frac{1}{2}\cot\frac{1}{2}x\, dx,$$

where C is a complete contour round the origin. This is

$$\frac{(-1)^{h-1}}{2\pi i}\int_{C} f^{(2h+1)}(x)\, \frac{1}{2}\cot\frac{1}{2}x\, dx = (-1)^h f^{(2h+1)}(0);$$

and this is accordingly the Abel-Poisson sum of the series $\Sigma n^{2h+1} b_n$.

A slightly modified form of the argument will prove that the k^{th} derived series of the Fourier series of $f(x)$ is summable (A) to $f^{(k)}(x)$ at any internal point of $(-\pi, \pi)$.[10] Suppose that

$$f(x) \sim \sum_{-\infty}^{\infty} c_n e^{nix},\ c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-nix} f(x)\, dx,$$

and that $-\pi < \xi < \pi$. Then we can express the sums

$$\sum_{n \geqq 0} c_n e^{ni\xi-\delta n},\ \sum_{n<0} c_n e^{ni\xi+\delta n} \qquad (\delta > 0)$$

by integrals along paths C_2, C_1 from $-\pi$ to π, C_1 above and C_2 below the real axis. The proof then follows the same lines.

It is also easy to show that the series are summable (B); we may prove this directly from the definitions, or use the theorem that a power series is summable (B) at any regular point on the circle of convergence.

[10] This was pointed out to me by Professor Pólya.

B. The series $\sum a^r \phi^{(r)}(x)$

3.1. The series of §2.2 were of the type

$$\sum_{r=0}^{\infty} a^r \phi^{(r)}(x),$$

and we expressed these sums by integrals involving $\phi(x)$ and exponential functions. A more entertaining form can be given to the results by considering series infinite in both directions. We write

$$\phi^{(-r)}(x) = \phi_r(x) = \frac{1}{(r-1)!}\int_0^x (x-t)^{r-1}\phi(t)\,dt \qquad (r>0),$$

so that a derivative of negative order $-r$ is a 'Riemann-Liouville' integral of order r and with origin 0; and we consider the series

$$y(x) = \sum_{-\infty}^{\infty} a^r \phi^{(r)}(x) = \sum_{-\infty}^{\infty} b^{-r}\phi^{(r)}(x) \qquad \left(b = \frac{1}{a}\right). \tag{3.1.1}$$

I suppose, in the first instance, that x and a are positive.

If the series (3.1.1) is uniformly convergent in any interval of x, then

$$y' = by,\ y = Ce^{bx}.$$

Hence we may expect the sum of (3.1.1) to be an exponential function, whatever the form of $\phi(x)$.[11]

We write

$$y = \sum b^{-r}\phi^{(r)}(x) = \sum_{r<0} + \sum_{r\geqq 0} = y_1 + y_2. \tag{3.1.2}$$

The first series may be summed at once, since

$$y_1 = \sum_{r>0} \frac{b^r}{(r-1)!}\int_0^x (x-t)^{r-1}\phi(t)dt = be^{bx}\int_0^x e^{-bt}\phi(t)\,dt. \tag{3.1.3}$$

For this no more is necessary than that $\phi(t)$ should be integrable. If we can prove that (with some interpretation of the series)

$$y_2 = \sum_{r\geqq 0} b^{-r}\phi^{(r)}(x) = b\int_0^{\infty} e^{-bu}\phi(x+u)\,du = be^{bx}\int_x^{\infty} e^{-bt}\phi(t)\,dt, \tag{3.1.4}$$

then (3.1.3) and (3.1.4) will give

$$y = Ce^{bx},\ C = b\int_0^{\infty} e^{-bt}\phi(t)\,dt. \tag{3.1.5}$$

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[11] See Pólya and Szegö, *Aufgabe aus der Analysis*, I (problems **165, 251,** pp. 30, 133, 185, 314). The formal remark is probably to be found in the work of Heaviside.

These formulae are certainly true, for example, if $\phi(x)$ is integral and of type less than b; in this case the series are convergent.

The work of §2.2 suggests that (3.1.4) should be true under considerably more general conditions when the series is interpreted as a Borel sum. The ordinary Borel sum is

$$\int_0^\infty e^{-t} \sum_0^\infty \frac{(at)^r}{r!} \phi^{(r)}(x)\, dx = \int_0^\infty e^{-t} \phi(x + at)\, dt = b \int_0^\infty e^{-bu} \phi(x + u)\, du$$

if (i) $\phi(x)$ is an integral function and (ii) the integrals are convergent. The conditions that the series should be summable (B^2) are more general. Suppose, for example, that $a > 0$, $x > 0$, and that $\phi(\xi + i\eta)$ is regular for $\xi > 0$. Then the series

$$\sum \frac{(at)^r}{r!} \phi^{(r)}(x) \tag{3.1.6}$$

has a positive radius of convergence, and $x + at$ lies, for every positive t, inside its Borel polygon of summability, so that the series is summable (B) to $\phi(x + at)$. Hence (3.1.4) *is true, in the* (B^2) *sense, whenever* (i) $\phi(\xi + i\eta)$ *is regular for* $\xi > 0$ *and* (ii) *the integral is convergent.*

For example, we may take

$$\phi(x) = \frac{x^c}{\Gamma(c)},$$

where $c > -1$: if $c = 0$, $\phi(x) = 0$. We thus obtain the formula

$$\sum_{-\infty}^{\infty} a^r \frac{x^{c-r}}{\Gamma(c - r)} = a^c e^{x/a}. \tag{3.1.7}$$

Here it is understood that terms for which $c - r$ is 0 or a negative integer are interpreted as 0.

The formula (3.1.7) is proved, in the first instance, for $x > 0$, $a > 0$, $c > -1$. These conditions may be relaxed. In the first place, replacing c by $c + 1, c + 2, \cdots$, we may remove the restriction on c. We may also suppose c complex. Finally we may extend the formula to complex values of x and a. For here $\phi(\xi + i\eta)$ is regular except at the origin, and not merely in the right half-plane. Hence, if

$$|\arg x - \arg a| < \tfrac{1}{2}\pi,$$

or $\Re(x/a) > 0$, $x + at$ lies, for all positive t, within the Borel polygon of (3.1.6). Hence (3.1.7) is true whenever $\Re(x/a) > 0$.

When c is an integer, the series reduces to the ordinary exponential series. The formula itself seems to be due substantially to Heaviside.[12]

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[12] See H. Jeffreys, *Operational methods in mathematical physics*, Camb. Math. Tracts, No. 23, p. 91.

Ingham and Jeffreys[13] have shown that Heaviside's formula is true asymptotically, that is to say that

$$\sum_R^\infty \frac{x^{c-r}}{\Gamma(c-r)} = e^x + O(x^{c-R-1})$$

for large positive x. It is natural to ask whether (3.1.4) and (3.1.5) are true in the same sense. We take $a = 1$.

If x is positive, and $\phi(\xi + i\eta)$ regular for $\xi > 0$, then

$$\sum_0^R t^r \frac{\phi^{(r)}(x)}{r!} = \frac{1}{2\pi i}\int_C \frac{\phi(u)}{u-x}\sum_0^R \left(\frac{t}{u-x}\right)^r du,$$

where C is a circle, of radius less than x, round $u = x$. We may replace C by the circle γ whose centre is $x + t$ and whose radius is $\frac{1}{2}x + t$. Then

$$\sum_0^R t^r \frac{\phi^{(r)}(x)}{r!} = \frac{1}{2\pi i}\int_\gamma \frac{\phi(u)}{u-x-t}\left\{1 - \left(\frac{t}{u-x}\right)^{R+1}\right\} du$$

$$= \phi(x+t) - \frac{1}{2\pi i}\int_\gamma \frac{\phi(u)}{u-x-t}\left(\frac{t}{u-x}\right)^{R+1} du = \phi(x+t) - J,$$

say.

Suppose now that $\phi(u) = O\,|\,u\,|^K)$, where $K > 0$, in the right hand half plane. Then, on γ, $|\,\phi(u)\,| < A(x^K + t^K)$, $|\,u - x - t\,| > Ax$, $|\,u - x\,| > Ax$, where the A are independent of u, x, t, R. Hence

$$|\,J\,| < A(x^K + t^K)\left(\frac{t}{x}\right)^{R+1} < A(x^{K-R-1}t^{R+1} + x^{-R-1}t^{K+R+1}),$$

$$\left|\int_0^\infty e^{-t} J\,dt\right| < A\{\Gamma(R+2)\,x^{K-R-1} + \Gamma(K+R+2)\,x^{-R-1}\}$$

$$< A\,\Gamma(K+R+2)\,x^{K-R-1},$$

$$\sum_0^R \phi^{(r)}(x) = \int_0^\infty e^{-t}\sum_0^R t^r \frac{\phi^{(r)}(x)}{r!}\,dt = \int_0^\infty e^{-t}\phi(x+t)\,dt + O(x^{K-R-1}),$$

for large x. In this sense the series is asymptotic. When, as in Heaviside's formula, $\phi(x)$ is a power of x, the series is asymptotic in the ordinary sense, the error after the term of order x^{c-R} being of order x^{c-R-1}.

C. The W, V, and B methods of summation of Fourier series

4.1. We may say that $\sum a_n$ is summable (A, α), where $\alpha > 0$, to sum s, if $\sum a_n e^{-\delta n^\alpha}$ is convergent for all positive δ and tends to s when $\delta \to 0$. The

[13] Jeffreys, *l. c.*

particular case which is relevant here is the case $\alpha = 2$. In this case it is convenient to replace δ by $(\delta\pi)^2$, so that

$$\sum a_n e^{-(n\delta\pi)^2} \to s\,. \tag{4.1.1}$$

In these circumstances I shall say, for reasons which will appear in a moment, that the series is summable (W).

A series may be summable (W) without being summable (A); thus

$$1 - x + x^2 - x^3 + \cdots$$

is summable (W), to sum $(1 + x)^{-1}$, for some $x > 1$. It is however easy to prove that *if* $\sum a_n$ *is summable* (W), *and* $\sum a_n e^{-ny}$ *is convergent for all positive* y, *then* $\sum a_n$ *is summable* (A).[14] In a word, the (W) method is more crudely powerful, but, for delicately divergent series, the ordinary Abel-Poisson method is always the more effective. In particular a Fourier series, if summable (W), is certainly summable (A).

It is not quite so easy to give an example of a Fourier series which is summable (A) and not summable (W). I have however shown[15] that the Dirichlet series

$$\sum n^{-s} e^{Ai(\log n)^2} \qquad (s > 0,\ A > 0)$$

is summable (A) if $s > 1 - A\pi$ and summable (W) if $s > 1 - \frac{1}{2}A\pi$, these inequalities being the best of their kind. Hence the series

$$\sum n^{-1+\delta} e^{Ai(\log n)^2} \cos nx\,,$$

with

$$0 < \delta < \tfrac{1}{2}\,,\ \tfrac{1}{2}A\pi < \delta < A\pi$$

is a Fourier series summable (A), but not summable (W), for $x = 0$.

If

$$\tfrac{1}{2}A_0 + \sum A_n(t) = \tfrac{1}{2}a_0 + \sum (a_n \cos nt + b_n \sin nt)$$

---

[14] From

$$e^{-ny} = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-n^2x^2y^2 - 1/x^2}\, \frac{dx}{x^2}$$

we deduce

$$\sum a_n e^{-ny} = \frac{2}{\sqrt{\pi}} \int_0^\infty f(xy) e^{-1/x^2}\, \frac{dx}{x^2}\,,$$

where $f(u) = \Sigma a_n e^{-(nu)^2}$, and the conclusion follows from the continuity of $f(u)$ at the origin. Much more general theorems concerning the relations of the methods (A, α) and (A, β) have been proved by M. L. Cartwright, *Proc. London Math. Soc.* (2), 31 (1930), 81–96.

[15] G. H. Hardy, 'The application of Abel's method of summation to Dirichlet's series', *Quarterly Journal*, 47 (1916), 176–192 (192).

is the Fourier series of $f(t)$, then

$$A_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos n(t-x)\,dt,$$

$$\frac{1}{2}A_0 + \sum A_n(x)\,e^{-(n\delta\pi)^2} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,K(t-x)\,dt,$$

where

$$K(u) = \frac{1}{2} + \sum_1^\infty \cos nu\; e^{-(n\delta\pi)^2} = \frac{1}{2\delta\sqrt{\pi}}\sum_{-\infty}^{\infty} e^{-(2n\pi-u)^2/4\pi^2\delta^2},$$

by a familiar formula of elliptic functions. It is plain that, when we substitute the second series for $K(t-x)$ under the sign of integration, all the terms are trivial except that for which $n = 0$. Hence, *in order that the series be summable* (W) *for* $t = x$, *it is necessary and sufficient that*

$$\frac{1}{2\delta\sqrt{\pi}}\int_{-\pi}^{\pi} f(t)\,e^{-(t-x)^2/4\pi^2\delta^2}\,dt \to s, \tag{4.1.2}$$

i.e. that Weierstrass's singular integral for $f(t)$ *should converge to* s.

The (W) method, like the (A) method, sums the series at any point of 'mean continuity'. Mean continuity is defined as follows. We write, as usual,

$$\phi(t) = \phi(t, x) = \tfrac{1}{2}\{f(x+t) + f(x-t) - 2s\},$$

with arbitrary s, and

$$\phi_1(t) = \int_0^t \phi(u)\,du,\ \phi_2(t) = \int_0^t \phi_1(u)\,du, \cdots;$$

and we say that $f(t)$ has mean continuity at x, or tends to s in mean at x, if

$$\phi_k(t) = o(t^k) \tag{4.1.3}$$

for some k.

The series will be summable (W) if

$$\frac{1}{\delta}\int \phi(t)\,e^{-t^2/4\pi^2\delta^2}\,dt = o(1), \tag{4.1.4}$$

where the integration is over a small fixed interval including the origin; or, what is the same thing, if

$$\frac{1}{\delta}\int \phi_k(t)\left(\frac{d}{dt}\right)^k (e^{-t^2/4\pi^2\delta^2})\,dt = o(1).$$

Now

$$\left(\frac{d}{dt}\right)^k e^{-t^2/4\pi^2\delta^2} = \delta^{-k}\,Q_k\left(\frac{t}{\delta}\right)e^{-t^2/4\pi^2\delta^2},$$

where $Q_k(u)$ is a polynomial of degree k in u; and

$$\frac{1}{\delta}\int\left(\frac{t}{\delta}\right)^k Q_k\left(\frac{t}{\delta}\right)e^{-t^2/4\pi^2\delta^2}\,dt$$

is bounded in δ for every k. From this it follows in the ordinary way that (4.1.3) implies (4.1.4).

4.2. Another method of summation which has the property just proved for the (W) method is that of de la Vallée-Poussin. We say that $\sum a_n$ is summable (V) to s if

$$a_0+\frac{n}{n+1}a_1+\frac{n(n-1)}{(n+1)(n+2)}a_2+\cdots+\frac{n(n-1)\cdots 1}{(n+1)(n+2)\cdots 2n}a_n\to s.$$

Since

$$\frac{1}{2}+\frac{n}{n+1}\cos t+\frac{n(n-1)}{(n+1)(n+2)}\cos 2t+\cdots=2^{2n-1}\frac{(n!)^2}{2n!}\left(\cos\frac{1}{2}t\right)^{2n}$$

and

$$\frac{2^{2n-1}\,(n\,!)^2}{(2n)\,!}\sim\tfrac{1}{2}\sqrt{n\pi},$$

a necessary and sufficient condition for the summability of the Fourier series to s is that

$$\frac{1}{2}\sqrt{\frac{n}{\pi}}\int_{-\pi}^{\pi}f(t)\left\{\cos\frac{1}{2}(t-x)\right\}^{2n}dt\to s,$$

or, what is the same thing, that

$$\sqrt{n}\int\phi(t)\left(\cos\frac{1}{2}t\right)^{2n}dt=o(1), \tag{4.2.1}$$

the integration being again over a small fixed interval including the origin.

The relations between the (W) and (V) methods are much closer than those between either of them and the Abel-Poisson method. In particular, *for Fourier series, the (W) and (V) methods are equivalent.*[16]

To prove this we have to show that the hypotheses (4.1.4) and (4.2.1) are equivalent. If we write $1/\nu$ for $\pi^2\delta^2$ in (4.1.4), it becomes

$$\sqrt{\nu}\int\phi(t)\,e^{-\frac{1}{4}\nu t^2}\,dt=o(1). \tag{4.2.2}$$

Here however ν tends to infinity continuously, whereas n is an integer in (4.2.1).

We therefore prove first that if

$$N<\nu=N+f<N+1, \qquad \pi^2\delta^2=\frac{1}{\nu}, \qquad \pi^2\delta_0^2=\frac{1}{N},$$

[16] Mr. J. M. Hyslop has shown that the two methods are equivalent for all series in which $a_n=O(n^k)$ for some k.

and $a_n = o(1)$, then

$$\sum a_n \{e^{-(n\delta\pi)^2} - e^{-(n\delta_0\pi)^2}\} \to 0 \tag{4.2.3}$$

when $N \to \infty$, uniformly in f. This will show that, when $a_n = o(1)$, (4.1.1), if true for the special sequence (δ_0), is true generally, so that we may suppose ν an integer in (4.2.2). For this, it is enough to prove

$$\sum \{e^{-n^2/(N+1)} - e^{-n^2/N}\}$$

bounded; and this sum is less than

$$\sum \{e^{-n/(N+1)} - e^{-n/N}\} = \frac{1}{1 - e^{-1/(N+1)}} - \frac{1}{1 - e^{-1/N}}$$

$$= N + 1 + O(1) - N + O(1) = O(1).$$

We may therefore replace ν, in (4.2.2), by n.

We choose a value of α between $\frac{3}{8}$ and $\frac{1}{2}$. If $t > n^{-\alpha}$ then

$$e^{-\frac{1}{4}n\,t^2} < e^{-\frac{1}{4}n^{1-2\alpha}}$$

and[17]

$$\left(\cos\frac{1}{2}t\right)^{2n} = e^{2n\log\cos\frac{1}{2}t} < e^{4n\log\left(1-\frac{1}{16}t^2\right)} < e^{-\frac{1}{4}n\,t^2} < e^{-\frac{1}{4}n^{1-2\alpha}}$$

Hence the parts of the integrals in which $t > n^{-\alpha}$ are negligible, and it is sufficient to prove that

$$J = \int_0^{n^{-\alpha}} |\phi(t)| \left\{\left(\cos\frac{1}{2}t\right)^{2n} - e^{-\frac{1}{4}n\,t^2}\right\} dt = o\left(n^{-\frac{1}{2}}\right).$$

Here

$$\left(\cos\frac{1}{2}t\right)^{2n} = e^{-\frac{1}{4}n\,t^2 + O(n\,t^4)},$$

$$\left(\cos\frac{1}{2}t\right)^{2n} - e^{-\frac{1}{4}n\,t^2} = O(nt^4),$$

and so

$$J = O\left\{n\int_0^{n^{-\alpha}} t^4\,|\phi(t)|\,dt\right\} = O(n^{1-4\alpha}) = o\left(n^{-\frac{1}{2}}\right),$$

which proves the equivalence of the two methods.

4.3. I conclude by a few remarks concerning the summability of Fourier series by Borel's method. This method is much less useful than those which

[17] $\cos t < 1 - \frac{1}{2}t^2 + \frac{1}{24}t^4 < \left(1 - \frac{1}{4}t^2\right)^2$.

we have been considering since §4.1, as has been shown by C. N. Moore[18] (and as will appear incidentally in the sequel) it is not always effective at a point of continuity. Littlewood and I have however shown[19] that Borel's and similar methods can be of service in the theory of convergence of Fourier series.

In what follows I suppose that $\phi(t)$ tends to 0 with t. Since a_n and b_n are $o(1)$, the various forms of Borel's definition are equivalent. I take the definition

$$s = \lim_{\xi\to\infty} e^{-\xi} \sum s_n \frac{\xi^n}{n!}.$$

Since

$$\sum \frac{\xi^n}{n!} \sin\left(n+\frac{1}{2}\right)t = e^{\xi\cos t} \sin\left(\xi \sin t + \frac{1}{2}t\right),$$

the series will be summable to s if and only if

$$\int \phi(t)\, e^{-\xi(1-\cos t)} \frac{\sin\left(\xi\sin t + \frac{1}{2}t\right)}{\sin\frac{1}{2}t}\, dt \to 0. \tag{4.3.1}$$

Here we may, on the usual grounds, omit the $\frac{1}{2}t$ in the argument of

$$\sin\left(\xi\sin t + \frac{1}{2}t\right),$$

and replace $\sin\frac{1}{2}t$ in the denominator by $\frac{1}{2}t$. The condition thus takes the form

$$\int \phi(t)\, e^{-\xi(1-\cos t)} \frac{\sin(\xi\sin t)}{t}\, dt \to 0. \tag{4.3.2}$$

The condition (4.3.2) may be simplified further. Since $1-\cos t > \frac{1}{3}t^2$ for small t, $\phi(t) = o(1)$, and

$$\int_{\xi^{-\frac{1}{2}}}^{\infty} \frac{e^{-\frac{1}{3}\xi t^2}}{t}\, dt = \int_1^{\infty} \frac{e^{-\frac{1}{3}u^2}}{u}\, du$$

is bounded, we may ignore the part of the integral for which $t > \xi^{-\frac{1}{2}}$. If $t < \xi^{-\frac{1}{2}}$ then

$$e^{-\xi(1-\cos t)} \sin(\xi\sin t) - \sin\xi t = O(\xi t^2)$$

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[18] *Proc. Nat. Acad. of Sci.*, 11 (1925), 284–287. See also E. Hille and J. D. Tamarkin, *Math. Annalen*, 108 (1933), p. 557.

[19] 'Some new convergence criteria for Fourier series', *Annali d. R. Sc. di Pisa* (2), 3 (1934), 43–62.

and

$$\int_0^{\xi^{-\frac{1}{2}}} |\phi(t)|\, O(\xi t)\, dt = o\left(\xi \int_0^{\xi^{-\frac{1}{2}}} t\, dt\right) = o(1).$$

Hence *a necessary and sufficient condition for summability*[20] *is that*

$$\int_0^{\xi^{-\frac{1}{2}}} \phi(t)\, \frac{\sin \xi t}{t}\, dt \to 0. \tag{4.3.3}$$

Suppose now that

$$\phi(t) = o\left\{\left(\log \frac{1}{t}\right)^{-1}\right\}. \tag{4.3.4}$$

Then

$$J(\xi) = \int_0^{\xi^{-\frac{1}{2}}} \phi(t)\, \frac{\sin \xi t}{t}\, dt = \int_0^{\xi^{-1}} + \int_{\xi^{-1}}^{\xi^{-\frac{1}{2}}} = J_1(\xi) + J_2(\xi).$$

Here

$$J_1(\xi) = o\left(\int_0^{\xi^{-1}} \xi\, dt\right) = o(1),$$

and

$$J_2(\xi) = o\left\{\int_{\xi^{-1}}^{\xi^{-\frac{1}{2}}} \frac{dt}{t \log (1/t)}\right\} = o(\log 2) = o(1).$$

Hence *the series is summable* (B), *to* s, *whenever* $\phi(t)$ *satisfies* (4.3.4). It is this result, in a different and generalised form, which Littlewood and I use in the paper just referred to. Combined with Tauberian theorems, it yields interesting criteria for ordinary convergence.

It is easy to deduce from (4.3.3) that Borel summation is not always effective at a point of continuity. The proofs that the Fourier series of a continuous function is not necessarily convergent depend, at bottom, on the fact that

$$\int_0^a \frac{\sin^2 \xi t}{t}\, dt$$

is not bounded. Since this is equally true of

$$\int_0^{\xi^{-\frac{1}{2}}} \frac{\sin^2 \xi t}{t}\, dt,$$

the same is true of Borel summability.

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[20] When $\phi(t) = o(1)$; in particular, when $f(t)$ is continuous at x.

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ON SUMMATION OF DERIVED SERIES OF THE CONJUGATE FOURIER SERIES

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1. **Introduction.** We have been unable to find in the literature a single theorem concerning the summability of the r^{th} derived series of the conjugate Fourier series for the case $r > 1$. Our principal result in this paper is such a theorem for the N_{z_p} summation method which we defined in an earlier paper.[1] Since the Riesz equivalent of the Cesàro method, $(C, p + \delta)$, $\delta > 0$, can be obtained by specialization of the kernel $z_p(t)$ of the N_{z_p} method, our theorem gives a new result for the Cesàro method.

2. **Notation.** The following notation will be used throughout the paper. We consider only functions $f(x)$ which are Lebesgue integrable $[f(x) \subset L]$ on $(-\pi, \pi)$ and periodic of period 2π and at times require that $\frac{d^{r-1}}{dx^{r-1}} f(x)$ be of bounded variation $\left[\frac{d^{r-1}}{dx^{r-1}} f(x) \subset B\right]$. We set

$$a_n \equiv 1/\pi \int_{-\pi}^{\pi} f(s) \cos ns \, ds, \; b_n \equiv 1/\pi \int_{-\pi}^{\pi} f(s) \sin ns \, ds .$$

Then $f(x)$ generates the Fourier series

$$a_0/2 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{2.1}$$

whose conjugate series is the series

$$\sum_{n=1}^{\infty} (-b_n \cos nx + a_n \sin nx) . \tag{2.2}$$

We designate by $\sigma_n(x)$ and $\sigma_n^{(r)}(x)$, respectively, the sums of the first n terms of (2.2) and of its r^{th} derived series. In connection with the N_{z_p} method we use the notation that we used in defining the method.[2] Then $N_{z_{r+1}} \sigma_n^{(r)}(x)$ represents the $N_{z_{r+1}}$ transform of $\sigma_n^{(r)}(x)$.

In defining functions to serve as sums for the derived series of the series (2.2) we use the notion of a generalized derivative in the sense of de la Vallée-Poussin[3] given by the following definition.

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[1] On the Nevanlinna and Bosanquet-Linfoot Summation Methods, These Annals, Vol. 35 (1934), p. 239–247.

[2] Loc. cit., note 1, §6.

[3] Ch. de la Vallée-Poussin, Sur l'approximation des functions d'une variable réelle, Acad. Bruxelles, Bulletin 1908.

DEFINITION 2.1. If at a point x the function $f(x)$ satisfies an equation of the form

$$\frac{f(x+s)+(-1)^r f(x-s)}{2} = \sum_{i=0}^{[r/2]} f^{(r-2i)}(x)\,\frac{s^{r-2i}}{(r-2i)!} + w(x,s)\,\frac{s^r}{r!}$$

where $w(x, s) \to 0$ as $s \to 0$, then $f^{(r)}(x)$ is the r^{th} generalized derivative of $f(x)$.

It follows from the existence of the r^{th} generalized derivative that the $(r-2i)^{\text{th}}$, $0 < 2i \leqq r$, also exist. When the ordinary derivative $d^r f(x)/ds^r$ exists it is identical with the generalized derivative $f^{(r)}(x)$. Throughout the paper we distinguish between ordinary and generalized derivatives in the manner just indicated.

We set

$$\Phi_r(s) \equiv \frac{(-1)^{r+1} f(x+s) + f(x-s)}{2}; \tag{2.3}$$

$$\Theta(s,n) \equiv \frac{\cos s/2 - \cos(n+1/2)s}{\sin s/2}; \tag{2.4}$$

$$A_r(s) \equiv 1/\pi\left[\Phi_r(s) + (-1)^r \sum_{i=0}^{\left[\frac{r-1}{2}\right]} \frac{s^{r-1-2i}}{(r-1-2i)!} f^{(r-1-2i)}(x)\right]; \tag{2.5}$$

$$A_r^*(s) \equiv \int_s^{1/n^2} A_r(s)\,\frac{d^r}{ds^r}\cot s/2\,ds; \tag{2.6}$$

$$B_r(n) \equiv (-1)^r \sum_{i=0}^{\left[\frac{r-1}{2}\right]} \frac{f^{(r-1-2i)}(x)}{\pi} \int_0^{\pi} \frac{s^{r-1-2i}}{(r-1-2i)!}\,\frac{d^r}{ds^r}\,\Theta(s,n)\,ds; \tag{2.7}$$

$$C_r \equiv \sum_{i=0}^{[r/2]-1} f^{(r-1-2i)}(x) \sum_{j=i}^{[r/2]-1} \frac{\pi^{2j-2i}}{(2j+1-2i)!}\,\frac{d^{2j+1}}{ds^{2j+1}}\cot s/2\Bigg]_{s=\pi}; \tag{2.8}$$

$$d(i,n) \equiv \sum_{j=0}^{i} \frac{(2i+1)!}{(2j+1)!\,(2i-2j)!}\,\frac{d^{2j+1}}{ds^{2j+1}}\cot s/2\,\frac{d^{2i-2j}}{ds^{2i-2j}}\cos ns - \frac{d^{2i+1}}{ds^{2i+1}}\sin ns\Bigg]_{s=\pi}; \tag{2.9}$$

$$D_r(n) \equiv \sum_{i=0}^{[r/2]-1} f^{(r-1-2i)}(x) \sum_{j=i}^{[r/2]-1} \frac{\pi^{2j-2i}}{(2j+1-2i)!}\,d(j,n); \tag{2.10}$$

and

$$F_i(ns) \equiv \int_0^1 z_{r+1}(t)\,t^i\,\frac{d^i}{dt^i}\cos nts\,dt,$$
$$G_i(ns) \equiv \int_0^1 z_{r+1}(t)\,t^i\,\frac{d^i}{dt^i}\sin nts\,dt. \tag{2.11}$$

In (2.7), (2.8), and (2.10) whenever the upper limit of a summation is less than the lower limit that summation must be interpreted as zero. Thus $B_0(n) = 0$ and $C_r = 0$, $D_r(n) = 0$ when $r = 0, 1$. We show later that $B_1(n) = 0$.

Using the notation just given we define $\tilde{f}^{(r)}(x)$ at a point where $f^{(r-1)}(x)$ exists by

DEFINITION 2.2. For $r = 0, 1, 2, \cdots$,

$$\tilde{f}^{(r)}(x) \equiv \lim_{\epsilon \to 0} \int_{\epsilon}^{\pi} A_r(s) \frac{d^r}{ds^r} \cot s/2 \, ds - C_r .$$

Our $\tilde{f}^{(0)}(x)$ is identical with $\tilde{f}(x)$ and our $\tilde{f}^{(1)}(x)$ is essentially the same as $\tilde{f}'(x)$ as these limits are usually defined.

3. **Expressions for $\sigma_n^{(r)}(x)$ and $N_{z_{r+1}}\sigma_n^{(r)}(x)$.** It is well known that

$$\sigma_n(x) = 1/2\pi \int_{-\pi}^{\pi} f(s) \, \Theta(x - s, n) \, ds . \tag{3.1}$$

Differentiating (3.1) r times we have, since $f(x)$ is of period 2π,

$$\begin{aligned} \sigma_n^{(r)}(x) &= 1/2\pi \int_0^{\pi} f(s) \frac{\partial^r}{\partial s^r} \Theta(x - s, n) \, ds \\ &= 1/2\pi \int_0^{\pi} [(-1)^{r+1} f(x + s) + f(x - s)] \frac{d^r}{ds^r} \Theta(s, n) \, ds . \end{aligned} \tag{3.2}$$

In obtaining from (3.2) the expression for $\sigma_n^{(r)}(x)$ which we use in this paper we refer to the following lemmas, proofs of which depend on elementary computations and hence are omitted.

LEMMA 3.1. *For n a positive integer*

$$\frac{\Theta(s, n)}{2} = \sin s + \sin 2s + \cdots + \sin ns ;$$

$$\frac{d^{2i}}{ds^{2i}} \Theta(s, n) = 0 \qquad \text{when} \qquad s = 0, \pi ;$$

$$\int_0^{\pi} \frac{d^{2i+1}}{ds^{2i+1}} \Theta(s, n) \, ds = 0 .$$

LEMMA 3.2. *When $0 \leqq s \leqq \pi$*

$$\frac{d^r}{ds^r} \cot s/2 = (-1)^r \left[(1/2)^r \, r! \cot^{r+1} s/2 + \sum_{i=0}^{\left[\frac{r-1}{2}\right]} a_i \cot^{r-1-2i} s/2 \right]$$

where the a's are positive constants: hence $\dfrac{d^{2i}}{ds^{2i}} \cot s/2 = 0$ when $s = \pi$.

LEMMA 3.3. $B_r(n) = C_r - D_r(n)$.

With the aid of Lemma 3.3 we obtain from (3.2) after adding and subtracting $B_r(n)$ the equation

$$\sigma_n^{(r)}(x) = \int_0^\pi A_r(s)\,\frac{d^r}{ds^r}\,\Theta(s, n)\,ds - C_r + D_r(n)\,. \tag{3.3}$$

We define the n^{th} partial sums of the r^{th} derived series of the series (2.2) for non-integral, as well as for integral, values of n by the right hand member of (3.3).

Then we have

$$\begin{aligned} N_{r+1}\,\sigma_n^{(r)}(x) &= \int_0^1 z_{r+1}(t)\,dt \int_0^\pi A_r(s)\,\frac{d^r}{ds^r}\,\Theta(s, nt)\,ds \\ &\quad - C_r + \int_0^\pi z_{r+1}(t)\,D_r(nt)\,dt = \int_{1/n^2}^\pi A_r(s)\,\frac{d^r}{ds^r}\cot s/2\,ds \\ &\quad - C_r + \int_0^1 z_{r+1}(t)\,dt \int_0^{1/n^2} A_r(s)\,\frac{d^r}{ds^r}\,\Theta(s, nt)\,ds \\ &\quad - \sum_{i=0}^{r} \frac{r!}{i!\,(r-i)!} \int_0^1 z_{r+1}(t)\,dt \int_{1/n^2}^\pi A_r(s)\,\frac{d^{r-i}}{ds^{r-i}}\cot s/2\,\frac{d^i}{ds^i}\cos nts\,ds \\ &\quad + \int_0^1 z_{r+1}(t)\,dt \int_{1/n^2}^\pi A_r(s)\,\frac{d^r}{ds^r}\sin nts\,ds + \int_0^1 z_{r+1}(t)\,D_r(nt)\,dt\,. \end{aligned} \tag{3.4}$$

Since

$$\frac{d^i}{ds^i}\begin{Bmatrix}\sin\\ \cos\end{Bmatrix} nts\Bigg\} = t^i/s^i\,\frac{d^i}{dt^i}\begin{Bmatrix}\sin\\ \cos\end{Bmatrix} nts\Bigg\}, \tag{3.5}$$

we have finally, after changing the order of integration in parts of (3.4),

$$\begin{aligned} N_{r+1}\,\sigma_n^{(r)}(x) &= \int_{1/n^2}^\pi A_r(s)\,\frac{d^r}{ds^r}\cot s/2\,ds - C_r \\ &\quad + \int_0^1 z_{r+1}(t)\,dt \int_0^{1/n^2} A_r(s)\,\frac{d^r}{ds^r}\,\Theta(s, nt)\,ds \\ &\quad - \sum_{i=0}^{r} \frac{r!}{i!\,(r-i)!}\left[\int_{1/n^2}^{\delta} + \int_{\delta}^{\pi}\right] A_r(s)/s^i\,\frac{d^{r-i}}{ds^{r-i}}\cot s/2\,F_i(ns)\,ds \\ &\quad + \left[\int_{1/n^2}^{\delta} + \int_{\delta}^{\pi}\right] A_r(s)/s^i\,G_r(ns)\,ds + \int_0^1 z_{r+1}(t)\,D_r(nt)\,dt \\ &\equiv \int_{1/n^2}^\pi A_r(s)\,\frac{d^r}{ds^r}\cot s/2\,ds - C_r + K_1 + K_2 + K_3 + K_4 + K_5 + K_6\,. \end{aligned} \tag{3.6}$$

4. **Preliminary lemmas.** We give here some rather simple lemmas concerning the function $\cot s/2$ and its derivatives. The proofs of these lemmas, which depend largely on Lemma 3.1, are obvious and hence are omitted.

LEMMA 4.1. *The function* $\cot s/2 = O(1/s)$, *and for* $i, j \geqq 0$, *the function*

$$\frac{\dfrac{d^i}{ds^i}\cot s/2}{\dfrac{d_j}{ds^j}\cot s/2} = O(s^{j-i}) \qquad \text{as} \qquad s \to 0\,.$$

LEMMA 4.2. *For* $r > 0$ *the function* $\dfrac{(-1)^r \cot^{r+1} s/2}{\dfrac{d^r}{ds^r}\cot s/2}$ *decreases steadily from* $2^r/r!$ *to* 0 *as* s *increases from* 0 *to* π.[4]

LEMMA 4.3. *For* $i > 0$ *the function* $1/(s^i \cot^i s/2)$ *increases steadily from* $(1/2)^i$ *to* ∞ *as* s *increases from* 0 *to* π.

LEMMA 4.4. *For* $r > 0$ *and* $0 \leqq i \leqq r$ *the function* $\dfrac{(-1)^{r-i}\dfrac{d^{r-i}}{ds^{r-i}}\cot s/2}{s^i \cot^{r+1} s/2}$ *increases steadily from* $(1/2)^r\,(r-i)\,!$ *to* ∞ *as* s *increases from* 0 *to* π.

LEMMA 4.5. *The function* $\dfrac{d^r}{ds^r}\Theta(s, nt) \Big/ \dfrac{d^r}{ds^r}\cot s/2 \to 0$ *as* $s \to 0$.

LEMMA 4.6. *The function* $\dfrac{d}{ds}\left[\dfrac{d^r}{ds^r}\Theta(s, nt) \Big/ \dfrac{d^r}{ds^r}\cot s/2\right]$ *is uniformly bounded with respect to* s, n, *and* t *provided* $n \geqq 0$, $0 \leqq s \leqq 1/n^2$, *and* $0 \leqq t \leqq 1$.

5. **Lemmas concerning certain limits.** From the definition of a non-absolutely convergent improper integral it follows that

LEMMA 5.1. *If* $f(x) \subset L$ *on* (ϵ, M) *for all* $\epsilon > 0$ *and* $\int_\alpha^\beta f(x)\,dx \to 0$ *as* $\alpha, \beta \to 0$ *then* $\int_0^M f(x)\,dx \equiv \lim_{\epsilon \to 0}\int_\epsilon^M f(x)\,dx$ *exists.*

LEMMA 5.2. *If* $g(x)$ *is monotone and bounded on* $(0, \epsilon)$, $\epsilon > 0$ *and*

$$\int_\alpha^\beta f(x)\,dx \to 0 \text{ as } \alpha, \beta \to 0 \text{ then } \int_\alpha^\beta f(x)\,g(x)\,dx \to 0 \text{ as } \alpha, \beta \to 0\,.$$

Proof. The second mean value theorem enables us to write

$$\int_\alpha^\beta f(x)\,g(x)\,dx = g(\alpha)\int_\alpha^\xi f(x)\,dx + g(\beta)\int_\xi^\beta f(x)\,dx \quad (\alpha \leqq \xi \leqq \beta)\,.$$

The lemma follows.

In the remaining lemmas of this paragraph we assume that $f^{(r)}(x)$ exists.

LEMMA 5.3. $\int_\alpha^\beta A_r(s)\dfrac{d^r}{ds^r}\cot s/2\,ds \to 0$ *as* $\alpha, \beta \to 0$.

[4] We use the word "steadily" to qualify the words "increasing" and "decreasing" when the function is never stationary.

LEMMA 5.4. *For* $0 \leqq i \leqq r+1$, $\int_\alpha^\beta A_r(s)\cot^i s/2\,ds \to 0$ *as* $\alpha, \beta \to 0$.

Proof. By Lemmas 4.2, 5.3, and 5.2

$$\int_\alpha^\beta A_r(s)\cot^{r+1} s/2\,ds = \int_\alpha^\beta A_r(s)\frac{d^r}{ds^r}\cot s/2 \cdot \frac{\cot^{r+1} s/2}{\frac{d^r}{ds^r}\cot s/2}\,ds \to 0$$

as $\alpha, \beta \to 0$; and then for $i \leqq r$,

$$\int_\alpha^\beta A_r(s)\cot^i s/2\,ds = \int_\alpha^\beta A_r(s)\cot^{r+1} s/2 \cdot \frac{1}{\cot^{r+1-i} s/2}\,ds \to 0 \text{ as } \alpha, \beta \to 0\,.$$

LEMMA 5.5. *For* $0 \leqq i \leqq r$, $\int_\alpha^\beta A_r(s)/s^i \frac{d^{r-i}}{ds^{r-i}}\cot s/2\,ds \to 0$ *as* $\alpha, \beta \to 0$.

Proof. By Lemmas 4.4, 5.4, and 5.2

$$\int_\alpha^\beta A_r(s)/s^i \frac{d^{r-i}}{ds^{r-i}}\cot s/2\,ds = \int_\alpha^\beta A_r(s)\cot^{r+1} s/2 \cdot \frac{\frac{d^{r-i}}{ds^{r-i}}\cot s/2}{s^i \cot^{r+1} s/2}\,ds \to 0 \text{ as } \alpha, \beta \to 0\,.$$

LEMMA 5.6. $\int_\alpha^\beta A_r(s)/s^r\,ds \to 0$ *as* $\alpha, \beta \to 0$.

Proof. By Lemmas 4.3, 5.4, and 5.2

$$\int_\alpha^\beta A_r(s)/s^r\,ds = \int_\alpha^\beta A_r(s)\cot^r s/2 \cdot \frac{1}{s^r \cot^r s/2}\,ds \to 0 \text{ as } \alpha, \beta \to 0\,.$$

6. **Lemmas arising from properties of** $z_{r+1}(t)$. The lemmas of this paragraph depend essentially on the fact that $z_{r+1}(t)$ is an $N_{z_{r+1}}$ kernel. We use the notation that we used in defining the N_{z_p} method.[5]

LEMMA 6.1. *For* $0 \leqq i < r$

$$\left|\int_0^1 t\frac{d^{i+1}}{dt^{i+1}}\left[t^i z_{r+1}(t)\right]\begin{matrix}\sin\\ \cos\end{matrix}\Big\} vt\,dt\right| \leqq K/v, \qquad (K = \text{const.})\,.$$

LEMMA 6.2.

$$\left|\int_0^1 t\frac{d^{r+1}}{dt^{r+1}}\left[t^r z_{r+1}(t)\right]\begin{matrix}\sin\\ \cos\end{matrix}\Big\} vt\,dt\right| \leqq K/v + Z_{r+1}(v)\,, \qquad (K = \text{const.})\,.$$

Proofs of Lemmas 6.1 and 6.2.[6] Since $z_{r+1}(t)$ is an $N_{z_{r+1}}$ kernel the functions

$$\frac{d^{i+1}}{dt^{i+1}}t^i z_{r+1}(t) \quad (i < r)\,, \qquad \frac{d^{r+1-j}}{dt^{r+1-j}}z_{r+1}(t) \quad (1 \leqq j \leqq r+1)\,,$$

[5] Loc. cit., note 1, §6.

[6] Throughout the proofs of lemmas of this paragraph sin may be replaced by cos.

are of bounded variation on (0, 1). Lemma 6.1 follows. Also

$$\left| \int_0^1 t \frac{d^{r+1}}{dt^{r+1}} [t^r z_{r+1}(t)] \sin vt \, dt \right|$$

$$\leqq \sum_{j=1}^{r+1} \frac{(r+1)!}{j!(r+1-j)!} \left| \int_0^1 t \frac{d^j}{dt^j} t^r \cdot \frac{d^{r+1-j}}{dt^{r+1-j}} z_{r+1}(t) \sin vt \, dt \right|$$

$$+ \left| \int_0^1 t^{r+1} \frac{d^{r+1}}{dt^{r+1}} z_{r+1}(t) \sin vt \, dt \right| \leqq K/v + Z_{r+1}(v) .$$

LEMMA 6.3. *For $i \leqq r$ the functions $F_i(v)$ and $G_i(v)$ are of bounded variation on $(0, \infty)$.*

Proof. Using Lemmas 6.1 and 6.2 and properties of $z_{r+1}(t)$ we see that the total variation of $G_i(v)$ on $(0, \infty)$ is

$$\int_0^\infty \left| \frac{d}{dv} \int_0^1 t^i z_{r+1}(t) \frac{d^i}{dt^i} \sin vt \, dt \right| dv$$

$$= \int_0^\infty \left| \frac{d}{dv} \int_0^1 \frac{d^i}{dt^i} [t^i z_{r+1}(t)] \sin vt \, dt \right| dv$$

$$= \int_0^\infty \left| \int_0^1 t \frac{d^i}{dt^i} [t^i z_{r+1}(t)] \cos vt \, dt \right| dv = \int_0^M + \int_M^\infty = O(1)$$

$$+ \int_M^\infty 1/v \left| \int_0^1 \left\{ \frac{d^i}{dt^i} [t^i z_{r+1}(t)] + t \frac{d^{i+1}}{dt^{i+1}} [t^i z_{r+1}(t)] \right\} \sin vt \, dt \right| dv$$

$$\leqq \begin{cases} O(1) + \int_M^\infty K/v^2 \, dv = O(1) & \text{when} \quad i < r \\ O(1) + \int_M^\infty K/v^2 + Z_{r+1}(v)/v \, dv = O(1) & \text{when} \quad i = r \end{cases} .$$

The proof for $F_i(v)$ is similar.

Lemma 6.3 enables us to set

$$\begin{aligned} F_i(ns) &= P_{i_1}(ns) - P_{i_2}(ns) , \\ G_i(ns) &= Q_{i_1}(ns) - Q_{i_2}(ns) , \end{aligned} \tag{6.1}$$

where the P's and Q's are positive, bounded, monotone decreasing functions of ns for $0 \leqq ns$. We note that the P's and Q's are uniformly bounded with respect to n and s, and for a fixed n become monotone decreasing functions of s for s on $(0, \infty)$.

LEMMA 6.4. *For $i \leqq r$*

$$| F_i(v) | \leqq K/v , \qquad | G_i(v) | \leqq K/v , \qquad (K = \text{const.}) .$$

Proof. In the proof of Lemma 6.3 we showed that

$$\left| G_i(v) \right| = \left| \int_0^1 \frac{d^i}{dt^i} [t^i z_{r+1}(t)] \sin vt \, dt \right| .$$

Since for $0 \leqq i \leqq r$, the function $\frac{d^i}{dt^i}[t^i z_{r+1}(t)]$ is of bounded variation on $(0, 1)$ the lemma follows for $G_i(v)$. The proof for $F_i(v)$ is similar.

LEMMA 6.5. *For* $0 \leqq i \leqq r$

$$\int_0^1 z_{r+1}(t) \frac{d^i}{ds^i} \begin{matrix} \sin \\ \cos \end{matrix} nts\, dt \Big]_{s=\pi} \to 0 \qquad as \qquad n \to \infty .$$

Proof. Using (3.5), properties of $z_{r+1}(t)$, and integrating by parts we have

$$\left| \int_0^1 z_{r+1}(t) \frac{d^i}{ds^i} \sin nts\, dt \right| = 1/\pi^i \left| \int_0^1 \frac{d^i}{dt^i}[t^i z_{r+1}(t)] \sin n\pi t\, dt \right|$$

$$\leqq K/n\pi \qquad (K = \text{const.}) .$$

7. Special lemmas for existence theorems. In this paragraph we give lemmas that we use in proving our theorems concerning the existence of $\tilde{f}^{(r)}(x)$.

LEMMA 7.1. *Wherever* $f^{(r+1)}(x)$ *exists* $A_r(s) \frac{d^r}{ds^r} \cot s/2 = O(1)$ *as* $s \to 0$.

Proof. Wherever $f^{(r+1)}(x)$ exists we see using Definition 2.1 to expand $\Phi_r(s)$ that

$$A_r(s) \frac{d^r}{ds^r} \cot s/2 = \frac{(-1)^{r+1}}{\pi} [f^{(r+1)}(x) + w(x, s)]\, s^{r+1} \frac{d^r}{ds^r} \cot s/2$$

and the lemma follows.

LEMMA 7.2. *For* $0 < 2v \leqq r$, $A_{r-2v}(s) \frac{d^{r-2v}}{ds^{r-2v}} \cot s/2 = O(1)$ *as* $s \to 0$ *wherever* $\tilde{f}^{(r)}(x)$ *exists.*

Proof. Wherever $\tilde{f}^{(r)}(x)$ exists, $f^{(r-1)}(x)$ exists and again using Definition 2.1 and expanding $\Phi_{r-2v}(s) = \Phi_{r-2}(s)$ we have

$$A_{r-2v}(s) \frac{d^{r-2v}}{ds^{r-2v}} \cot s/2 = \frac{(-1)^{r-1}}{\pi} \left[\sum_{i=0}^{v-1} \frac{s^{r-1-2i}}{(r-1-2i)!} f^{(r-1-2i)}(x) \right.$$

$$\left. + w(x, s) \frac{s^{r-i}}{(r-1)!} \right] \frac{d^{r-2v}}{ds^{r-2v}} \cot s/2 = O\left[s^{r+1-2v} \frac{d^{r-2v}}{ds^{r-2v}} \cot s/2 \right] = O(1) \text{ as } s \to 0 .$$

In the statements and proofs of Lemmas 7.3 to 7.8 (inclusive) we assume, as everywhere in this paper, that $f(x) \subset L$ and is of period 2π and in addition that $f(x)$ is such that $\frac{d^{r-1}}{dx^{r-1}} f(x) \subset B$. Then $\frac{d^r}{dx^r} f(x)$ exists almost everywhere and, taken where it exists, belongs to L; and all generalized derivatives used in the definition of $\tilde{f}^{(r)}(x)$ become ordinary derivatives and are of period 2π.

LEMMA 7.3. *For* $0 < 2i \leqq r$, $\frac{d^{r-2i}}{ds^{r-2i}} \Phi_r(s) = 0$ *when* $s = \pi$.

LEMMA 7.4. $\sum_{j=0}^{r-1} (-1)^j \frac{d^j}{ds^j} A_r(s) \frac{d^{r-1-j}}{ds^{r-1-j}} \cot s/2 \Big]_{s=\pi} = C_r .$

Proof. Since by Lemma 3.2 the derivatives of even order of $\cot s/2$ are zero when $s = \pi$, we set $r - 1 - j = 2\alpha + 1$. $\sum_{j=0}^{r-1}$ then becomes $\sum_{\alpha=0}^{[r/2]-1}$. The lemma follows upon using Lemma 7.3, differentiating the portion of $A_r(s)$ which does not vanish, and changing the order of summation.

LEMMA 7.5. *Wherever* $\frac{d^r}{dx^r} f(x)$ *exists*[7]

$$f(x + s) = \sum_{i=0}^{r} \frac{s^i}{i!} \frac{d^i}{ds^i} f(x) + v(s, x)$$

where, as $s \to 0$,

$$\frac{\partial^i v(s, x)}{\partial s^i} = o(s^{r-i}), \qquad i = 0, 1, 2, \cdots, r - 1.$$

LEMMA 7.6. *Wherever* $\frac{d^r}{dx^r} f(x)$ *exists, we have as* $s \to 0$,

$$\frac{d^i}{ds^i} A_r(s) = o(s^{r-i}), \qquad i = 0, 1, \cdots, r - 1.$$

Proof. Follows upon using Lemma 7.5 to expand $\Phi_r(s)$.

LEMMA 7.7. *When* $0 \leqq j < r$, $\frac{d^j}{ds^j} A_r(s) \frac{d^{r-1-j}}{ds^{r-1-j}} \cot s/2 \to 0$ *as* $s \to 0$ *wherever* $\frac{d^r}{dx^r} f(x)$ *exists.*

Proof. Follows immediately from Lemmas 3.2 and 7.6.

LEMMA 7.8. *The limit*

$$\lim_{\epsilon \to 0} \int_{\epsilon}^{\pi} \frac{d^r}{ds^r} \Phi_r(s) \cot s/2 \, ds$$

exists almost everywhere.

Proof. Taken wherever it exists $\frac{d^r}{dx^r} f(x) \subset L$, also

$$\frac{d^r}{ds^r} \Phi_r(s) = (-1)^r \left[\frac{d^r}{dx^r} f(x + s) - \frac{d^r}{dx^r} f(x - s) \right].$$

The lemma follows from Plessner's lemma concerning the existence of $\tilde{f}(x)$.[8]

8. **Final lemmas.** Our summability theorem follows immediately from the following lemmas. We use here the notation of (3.6).

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[7] E. W. Hobson, The Theory of Functions of a Real Variable, Vol. I (3rd Edition), pp. 369–370. Since $\frac{d^{r-1}}{dx^{r-1}} f(x) \subset B$, the lemma follows from Hobson's results.

[8] A. Plessner, Zur Theorie der konjugierten trigonometrischen Reihen, Mitteilungen des Mathematischen Seminars der Universität Giessen, Heft 10, (1923).

LEMMA 8.1. $K_1 \to 0$ *as* $n \to \infty$.

Proof. Integrating by parts and using Lemmas 4.5, 4.6, and 5.3 we have

$$K_1 = \int_0^1 z_{r+1}(t)\,dt \int_0^{1/n^2} A_r(s)\frac{d^r}{ds^r}\cot s/2 \cdot \frac{d^r}{ds^r}\Theta(s,nt) \Big/ \frac{d^r}{ds^r}\cot s/2\,ds$$

$$= \int_0^1 z_{r+1}(t)\,dt \left\{ -A_r^*(s)\frac{d^r}{ds^r}\Theta(s,nt) \Big/ \frac{d^r}{ds^r}\cot s/2 \right]_0^{1/n^2}$$

$$+ \int_0^{1/n^2} A_r^*(s)\frac{d}{ds}\left\{ \frac{d^r}{ds^r}\Theta(s,nt) \Big/ \frac{d^r}{ds^r}\cot s/2 \right\} ds \Bigg\}$$

$$= [o(1) \text{ as } n \to \infty] \int_0^1 z_{r+1}(t)\,dt = o(1).$$

LEMMA 8.2. K_2 *and* $K_4 \to 0$ *as* $\delta \to 0$ *and* $n \to \infty$.

Proof. By (6.1), Lemma 5.5, and the second mean value theorem we have for $i \leqq r$

$$\int_{1/n^2}^{\delta} A_r(s)/s^i \frac{d^{r-i}}{ds^{r-i}}\cot s/2\, F_i(ns)\,ds$$

$$= \int_{1/n^2}^{\delta} A_r(s)/s^i \frac{d^{r-i}}{ds^{r-i}}\cot s/2 \left\{ P_{i_1}(ns) - P_{i_2}(ns) \right\} ds$$

$$= P_{i_1}(1/n) \int_{1/n^2}^{\alpha_1} A_r(s)/s^i \frac{d^{r-i}}{ds^{r-i}}\cot s/2\,ds$$

$$- P_{i_2}(1/n) \int_{1/n^2}^{\alpha_2} A_r(s)/s^i \frac{d^{r-i}}{ds^{r-i}}\cot s/2\,ds \qquad (1/n^2 \leqq \alpha_1, \alpha_2 \leqq \delta)$$

$$= o(1) \text{ as } \delta \to 0 \text{ and } n \to \infty.$$

The lemma follows since K_2 is a finite sum of terms of the kind treated above and K_4 can be handled in a similar manner.

LEMMA 8.3. K_3 *and* $K_5 \to 0$ *as* $n \to \infty$ *for a fixed* δ.

Proof. By Lemma 6.4, for $0 \leqq i \leqq r$,

$$\left| \int_{\delta}^{\pi} A_r(s)/s^i \frac{d^{r-i}}{ds^{r-i}}\cot s/2\, F_i(ns)\,ds \right|$$

$$\leqq K/n\delta \int_{\delta}^{\pi} \left| A_r(s)/s^i \frac{d^{r-i}}{ds^{r-i}}\cot s/2 \right| ds \to 0 \text{ as } n \to \infty \text{ for a fixed } \delta.$$

The lemma follows since K_3 is a finite sum of terms of the type considered above and K_5 can be treated in a similar manner.

LEMMA 8.4. $K_6 \to 0$ *as* $n \to \infty$.

Proof. Follows from Lemma 6.5 since K_6 is a finite sum of terms of the type treated in that lemma.

9. **Existence theorems for $\tilde{f}^{(r)}(x)$.** In the theorems given here we give sufficient conditions for the existence of $\tilde{f}^{(r)}(x)$.

THEOREM 9.1. *If $f(x) \subset L$ on $(-\pi, \pi)$ and is of period 2π, then $\bar{f}^{(r)}(x)$ exists wherever the generalized derivative $f^{(r+1)}(x)$ exists and consequently wherever $\frac{d^{r+1}}{dx^{r+1}} f(x)$ exists.*

Proof. It follows from Lemma 7.1 that the limit involved in the definition of $\bar{f}^{(r)}(x)$ exists as an ordinary L integral taken over $(0, \pi)$.

THEOREM 9.2. *Wherever $\bar{f}^{(r)}(x)$ exists the limits $\bar{f}^{(r-2v)}(x)$, $(0 < 2v \leqq r)$, also exist.*

Proof. As a consequence of Lemma 7.2 the limit involved in the definition of $\bar{f}^{(r-2v)}(x)$ exists as an ordinary L integral taken over $(0, \pi)$.

THEOREM 9.3. [Principal existence theorem.] *If $f(x) \subset L$ on $(-\pi, \pi)$, is of period 2π, and is such that $\frac{d^{r-1}}{dx^{r-1}} f(x) \subset B$ then*

$$\bar{f}^{(r)}(x) \equiv \lim_{\epsilon \to 0} (-1)^r \int_\epsilon^\pi \frac{d^r}{ds^r} \Phi_r(s) \cot s/2 \, ds$$

and exists almost everywhere.

Proof. Integrating by parts we have

$$\int_\epsilon^\pi A_r(s) \frac{d^r}{ds^r} \cot s/2 \, ds = \sum_{j=0}^{r-1} (-1)^j \frac{d^j}{ds^j} A_r(s) \frac{d^{r-1-j}}{ds^{r-1-j}} \cot s/2 \Big]_\epsilon^\pi + (-1)^r/\pi \int_\epsilon^\pi \frac{d^r}{ds^r} \Phi_r(s) \cot s/2 \, ds.$$

By Lemmas 7.4, 7.7, and 7.8

$$\bar{f}^{(r)}(x) \equiv \lim_{\epsilon \to 0} (-1)^r/\pi \int_\epsilon^\pi \frac{d^r}{ds^r} \Phi_r(s) \cot s/2 \, ds$$

and exists almost everywhere.

10. **Summability theorem.** The following theorem gives our principal result.

THEOREM 10.1. *If $f(x) \subset L$ on $(-\pi, \pi)$ and is periodic of period 2π then the N_{z_p} sum of the r^{th}, $r = 0, 1, 2, \cdots, p-1$, derived series of the conjugate Fourier series is $\bar{f}^{(r)}(x)$ wherever $\bar{f}^{(r)}(x)$ exists.*

Proof. It follows from (3.6) and Lemmas 8.1, 8.2, 8.3, and 8.4 that wherever $\bar{f}^{(r)}(x)$ exists

$$N_{z_{r+1}} \sigma_n^{(r)}(x) \to \bar{f}^{(r)}(x) \text{ as } n \to \infty ;$$

and then, since every N_{z_p} kernel is an N_{z_r} kernel, $r < p$,[9] it follows that when $p > r$

$$N_{z_p} \sigma_n^{(r)}(x) \to \bar{f}^{(r)}(x) \text{ as } n \to \infty .$$

The theorem follows.

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[9] Loc. cit., note 1, § 6.

Since choice of $z_p(t) = (p + \delta)\,(1 - t)^{p+\delta-1}$, $\delta > 0$, as kernel reduces the N_{z_p} method to the Riesz equivalent of the Cesàro method, $(C, p + \delta)$, Theorem 10.1 contains a new result for the Cesàro method.[10]

We plan to give additional theorems concerning the summability of the r^{th} derived conjugate series in a later paper.

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[10] This result is the best of its kind obtainable, for R. E. A. C. Paley in his paper, "On the Cesàro Summability of Fourier Series and Allied Series," Proceedings of the Cambridge Philosophical Society, Vol. 26 (1930), pp. 173–203, shows that the conjugate Fourier series is summable $(C, 1 + \delta)$, $\delta > 0$, but not necessarily $(C, 1)$ to $\tilde{f}(x)$ wherever that limit exists.

The index of summability in our theorem can be lowered if $f(x)$ is such that

$$\frac{d^{r-1}}{dx^{r-1}} f(x) \subset B \text{ on } (-\pi, \pi).$$

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SUR LA DIMENSION DES PRODUITS CARTÉSIENS

PAR W. HUREWICZ (Amsterdam)

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On ne connait que très peu de résultats généraux sur les proprietés géométriques de l'espace-produit $X \times Y$,[1] des espaces topologiques X et Y.[2] En particulier, en ce qui concerne la dimension[3] de l'espace $X \times Y$, il est comme on sait très facile de montrer

$$\dim (X \times Y) \leqq \dim X + \dim Y ,^{4} \tag{1}$$

tandis que la relation "logarithmique"

$$\dim (X \times Y) = \dim X + \dim Y \tag{2}$$

(qui a priori semble bien probable) est en général en défaut, même pour les espaces compacts, comme l'a prouvé M. Pontrjagin[5] en construisant deux espaces compacts de dimension 2, dont le produit Cartésien est de dimension 3. Le but du memoire présent est de démontrer que *la relation* (2) *est valable dans le cas particulier, où, X étant un espace compact de dimension arbitraire, Y est un espace séparable (non nécessairement compact) de dimension* 1.[6] Il en résulte immédiatement que d'une manière plus générale la relation (2) est encore juste, lorsque X est un espace compact quelconque et Y est produit Cartésien d'un nombre fini d'espaces de dimension 1.[7]

En tenant compte de la relation (1) triviale et du fait que tout espace de dimension positive finie contient un sous-ensemble de dimension 1, on voit facilement que la proposition que nous venons de formuler équivaut au théorème suivant:

THÉORÈME. *X étant un espace compact et Y étant un espace séparable de dimension positive, on a toujours:*

$$\dim (X \times Y) > \dim X .^{7a} \tag{3}$$

---

[1] On entend par "l'espace-produit" (ou bien "produit Cartésien") $X \times Y$ l'espace formé par les couples (x, y), x étant un élément de X, y—de Y, et les voisinages des couples étant définies de la façon connue. L'expression "Produit Cartésien est due à M. C. Kuratowski, voir son livre "Topologie I" (1933), p. 7 et 79.

[2] Dans cet ouvrage tous les espaces sont supposés *métrisables*.

[3] La notion de la dimension est prise dans le sens habituel de M. M. Brouwer, Menger et Urysohn.

[4] Voir p. ex. C. Kuratowski, l.c., p. 141.

[5] C. R. 190 (1930), p. 1105–1107.

[6] La validité de la relation (2) pour dim $Y = 0$ résulte immédiatement de (1).

[7] En particulier, le produit d'un nombre fini n d'espaces compacts de dimension 1 est toujours de dimension n. C'est un théorème que j'ai établi dans un mémoire antérieur. Nous y reviendrons plus loin (voir[13]).

[7a] Dans la démonstration on peut supposer que X est de dimension finie, car autrement la relation (3) devient triviale.

La démonstration s'appuie sur un résultat important de M. Alexandroff. Avant de formuler ce résultat rappelons quelques définitions fondamentales: X étant un espace compact, une *représentation continue de X sur la surface sphérique* à n dimensions S_n est dite *essentielle*, s'il n'est pas possible de la ramener par une modification continue à une représentation "constante," qui fait correspondre un point fixe de S_n à tous les points de X.[8] Un espace compact de dimension finie n s'appelle *cyclique*, lorsqu'il existe une représentation continue essentielle de X sur S_n.[9] Cela posé, le théorème de M. Alexandroff[10] peut être énoncé de la façon suivante:

X étant un espace compact de dimension finie, *il existe toujours un espace cyclique X' de même dimension qu'on obtient de l'espace X, en y remplaçant un ensemble fermé F par un point p*; d'une façon plus précise X' s'obtient de X par une transformation continue qui fait correspondre un point fixe p de X' à tous les points contenus dans F, les images de tous les autres points étant distincts de p et distincts deux à deux (la transformation considerée est donc une homéomorphie sur $X - F$). Si l'on considère maintenant le produit Cartésien $X' \times Y$, Y étant un espace quelconque, on aura

$$X' \times Y = (X' - p) \times Y + p \times Y .$$

Le second membre $p \times Y$ est un ensemble fermé dans la somme $X' \times Y$, et sa dimension ($= \dim Y$) est évidemment au plus égale à celle du premier membre; il en résulte d'après le théorème bien connu sur la dimension des ensembles-sommes:

$$\dim (X' \times Y) = \dim [(X' - p) \times Y] .$$

Or, $X' - p$ étant homéomorphe à un sous-ensemble de X, l'ensemble

$$(X' - p) \times Y$$

est une image homéomorphe d'une partie de l'espace $X \times Y$, et il s'ensuit de la rélation précedente:

$$\dim (X' \times Y) \leqq \dim (X \times Y) .$$

---

[8] Rappelons encore que, f et F étant deux représentations continues du même espace X sur S_n, elles se déduisent l'une de l'autre par une déformation continue, pourvu que les points-images $f(x)$ et $F(x)$ *ne soient antipodes* pour aucun $x \in X$. En effet, cette condition étant remplie, on peut transporter les points images de $f(x)$ en $F(x)$ (ou inversement) en les faisant décrire les arcs géodesiques joignant ces points. Il en résulte que, f étant une représentation essentielle (resp. non essentielle) de X sur S_n, *toute représentation continue de X sur S_n suffisamment voisine de f jouit de la même propriété.*

[9] M. Alexandroff (voir Math. Ann 106, p. 218) donne une autre définition des espaces cycliques et démontre ensuite (l.c. p. 223, Hauptsatz) qu'elle est équivalente avec celle que nous venons de formuler.

[10] Voir P. Alexandroff, l.c. p. 220.

Si l'on a donc dim $(X' \times Y) \geqq \dim X$, on aura à plus forte raison dim $(X \times Y) \geqq \dim X$; par conséquent *il suffit d'établir la rélation* (3) *pour le cas, où* X *est un espace cyclique*, ce que nous allons supposer dans la suite.

Soit donc X un espace compact cyclique de dimension finie n, et soit Y un espace séparable de dimension positive. Supposons qu'on ait:

$$\dim (X \times Y) = n . \tag{4}$$

La dimension de Y étant différente de zéro, on peut trouver dans Y deux ensembles disjoints fermés A et B, *tels qu'il soit impossible de décomposer* Y *en deux ensembles disjoints fermés, dont l'un contienne* A *l'autre* B. Soit maintenant f une représentation continue essentielle de X sur S_n. Nous définissons une représentation continue F de l'ensemble

$$M = X \times (A + B)$$

en posant pour $x \in X$, $y \in A$

$$F(x, y) = f(x)$$

et pour $x \in X$, $y \in B$:

$$F(x, y) = c = \text{const},$$

c étant un point quelconque déterminé de S_n.

Nous allons *prolonger la représentation* F *dans l'espace* $X \times Y$ *tout entier*, ce qui est possible en vertu le théorème suivant:[11] Soit E un espace séparable de dimension n, M—un ensemble fermé dans E, φ—une représentation continue de M sur S_n. Il existe alors une représentation continue de l'espace entier E sur S, coincidant avec φ dans tous les points de M. L'hypothèse (4) permet d'appliquer ce théorème à l'espace $X \times Y$ et à la fonction F. Il existe donc une représentation continue θ de l'espace $X \times Y$ sur S_n, coïncidant dans M avec la représentation F. Désignons pour $y \in Y$ par θ_y la représentation de X, définie par l'équation:

$$\theta_y(x) = \theta(x, y)$$

et considérons l'ensemble P, formé des *points* $y \in Y$, *pour lesquels* θ_y *est une représentation essentielle*. D'après la définition de la fonction θ on a:

$$P \supset A , \qquad Y - P \supset B .$$

D'autre part, il est facile de voir que les ensembles P et $Y - P$ sont tous les deux *fermés*: en effet, y étant un point fixe de Y et y_m $(m = 1, 2 \cdots)$ une suite de points convergeants vers y, les fonctions θ_{y_n} convergent uniformément vers

[11] Pour les espaces compacts c'est une conséquence presque immédiate d'un théorème de M. Alexandroff (l.c. p. 170). Une démonstration plus simple et valable pour les espaces séparables généraux se trouve dans mon mémoire "Über Abbildungen topologischer Räume auf die n-dimensionale Sphäre" (à paraître dans Fund. Math. 24).

la fonction θ_y,[12] par conséquent, si la représentation θ_y est essentielle (resp. non essentielle), pour n suffisamment grand la fonction θ_{y_n} l'est également.[8]

Les ensembles P et $Y - P$ jouissent donc des proprietés qui ont été exclues par le choix des ensembles A et B. L'hypothèse (4) conduit ainsi à une contradiction, et notre théorème est démontré.

Citons une de ses conséquences immédiates:

(a) *Le produit Cartésien d'un nombre fini n de continus compacts à une dimension égale ou superieure à n.*

C'est une proposition que j'ai établie dans un mémoire antérieur[13] et qui est beaucoup moins profonde que le théorème général. On la prouve directement d'une manière assez simple[13] sans utiliser le théorème difficile de M. Alexandroff qui a joué un rôle si important dans le raisonnement précédent.

Une autre conséquence immédiate de notre théorème est la proposition suivante:

X étant un espace compact de dimension finie, Y un espace séparable de dimension positive, l'espace produit $X \times Y$ n'est homéomorphe à aucun sous-ensemble de X.[14] Il n'est pas sans intérêt de remarquer que pour le cas, où Y est compact cette proposition se déduit aussi très facilement de la proposition (a). En effet, si l'espace $X \times Y$ était homéomorphe à une partie de X, les espaces

$$X \times Y^n \qquad (n = 1, 2 \cdots)^{15}$$

le seraient également, et il en résulterait $\dim (X \times Y^n) \leqq \dim X$ et à plus forte raison

$$\dim Y^n \leqq \dim X \qquad (n = 1, 2 \cdots),$$

ce qui est incompatible avec la proposition (a), puisque tout ensemble compact de dimension non nulle contient un continu, et par conséquent $\dim Y^n \geqq n$. Rémarquons que cette dernière preuve ne suppose pas que X soit compact ou même séparable.

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[12] Car, dans le cas contraire on pourrait supposer qu'il existe pour chaque indice m un point $x_m \in X$, en sorte que la distance entre les points images $\theta(x_m, y_o)$ et $\theta(x_m, y_m)$ soit superieure à un nombre fixe $r > 0$ (independant de m De plus, en utilisant la compacticité de X on peut admettre que les points x_n covergent vers un point limite x_o, ce qui nous amène à une contradiction évidente avec la continuité de la fonction $\theta(x, y)$.

[13] Voir Math. Ann. 102, p. 305–312.—Le théorème cité plus haut sous 7 n'est qu'une autre forme de la proposition (a).

[14] En se servant de la terminologie de M. Frechet on peut dire que l'espace $X \times Y$ a un type de dimension superieur à celui de l'espace X.

[15] Nous désignons par Y^n le produit Cartésien de n espaces homéomorphes à Y.

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TABLES OF IRREDUCIBLE POLYNOMIALS FOR THE FIRST FOUR PRIME MODULI

BY RANDOLPH CHURCH

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There seem to be no complete lists giving irreducible polynomials for prime moduli. Dickson[1] gives an irreducible polynomial for each degree within certain limits for the moduli 2, 3, 5, 7, 11. Arnoux[2] lists the decomposition of all polynomials of degree less than or equal to three for the modulus 5. More complete and more systematic lists do not seem to exist. The following tables were constructed to satisfy this need.

The notation of detached coefficients is used throughout; that is, the polynomial

$$a_0x^n + a_1x^{n-1} + \cdots + a_n$$

is written simply

$$a_0a_1a_2 \cdots a_{n-1}a_n .$$

In the tables, the polynomials are primary; that is, $a_0 = 1$. The left hand columns list all irreducible polynomials for the degree and modulus concerned. Each table is headed by the appropriate value of n, indicating the degree of the polynomials; naturally this is one less than the number of digits in the representation. The right hand columns, headed by e, contain the exponents to which the polynomials belong. For each irreducible polynomial, $\varphi(x)$, this exponent is the least integer e such that

$$x^e - 1 \equiv 0 \qquad (\text{mod } p, \varphi(x)) .$$

It will be recalled that those integers e occur as exponents, and only those, which have the property that

$$p^n - 1 \equiv 0 \qquad (\text{mod } e)$$

holds for n equal to the degree of $\varphi(x)$ but for no smaller power of p. The extent of the tables may be summarized:

for $p = 2$ up to and including $n = 11$,
for $p = 3$ up to and including $n = 7$,
for $p = 5$ up to and including $n = 5$,
for $p = 7$ up to and including $n = 4$.

[1] L. E. Dickson, Linear Groups, Teubner, Leipzig, 1901, p. 44.

[2] G. Arnoux, Essais de psychologie et métaphysique positive, Assoc. Franç., 31 (1902), pp. 202–227.

The irreducible polynomials for each degree were obtained by listing all polynomials of that degree, forming from those of lower degree the composite polynomials, and retaining those not occurring on the second list. The exponents were obtained by dividing $\varphi(x)$ into x^e, with increasing e, until the residue 1 was obtained. This work was shortened by the knowledge of the exponents to which the primary polynomials corresponding to $\varphi(g^a x)$ and $\varphi(1/x)$ belong when the exponent of $\varphi(x)$ is known, g^a being any power of a primitive root, modulo p.

Every precaution has been taken to insure the accuracy of the tables. Naturally the number of polynomials belonging to each exponent e was checked against the formula for this number: $\psi_n(e) = \varphi(e)/n$. It can easily be shown that if $\varphi(x)$ is irreducible of degree n and belongs to the exponent e, and if it has the final coefficient a_n, then

$$(-1)^n a_n = g^{\frac{p^n-1}{e}}$$

where g is a primitive root. This also provided a partial check. The determination of the exponents provided in itself a very satisfactory control of the irreducibility of the polynomials. Finally, for nearly all exponents, the polynomials belonging to them were multiplied together to give the known expression for this product. In the few cases where this was not done, linear transformation checks, or checks depending on theorems to be proved elsewhere, were applied.

IRREDUCIBLE POLYNOMIALS FOR THE MODULUS 2

$n = 1$	e
10	
11	1

$n = 2$	e
111	3

$n = 3$	e
1011	7
1101	7

$n = 4$	e
10011	15
11001	15
11111	5

$n = 5$	e
100101	31
101001	31
101111	31
110111	31
111011	31
111101	31

$n = 6$	e
1000011	63
1001001	9
1010111	21
1011011	63
1100001	63
1100111	63
1101101	63
1110011	63
1110101	21

$n = 7$	e
10000011	127
10001001	127
10001111	127
10010001	127
10011101	127
10100111	127
10101011	127
10111001	127
10111111	127
11000001	127
11001011	127
11010011	127
11010101	127
11100101	127
11101111	127
11110001	127
11110111	127
11111101	127

$n = 8$	e
100011011	51
100011101	255
100101011	255
100101101	255
100111001	17
100111111	85
101001101	255
101011111	255
101100011	255
101100101	255
101101001	255
101110001	255
101110111	85
101111011	85
110000111	255
110001011	85
110001101	255
110011111	51
110100011	85
110101001	255
110110001	51
110111101	85
111000011	255
111001111	255
111010111	17
111011101	85
111100111	255
111110011	51
111110101	255
111111001	85

$n = 9$	e
1000000011	73
1000010001	511
1000010111	73
1000011011	511
1000100001	511
1000101101	511
1000110011	511

IRREDUCIBLE POLYNOMIALS FOR THE MODULUS 2—*Continued*

		$n = 10$	e
1001001011	73		
1001011001	511		
1001011111	511	10000001001	1023
1001100101	73	10000001111	341
1001101001	511	10000011011	1023
1001101111	511	10000011101	341
1001110111	511	10000100111	1023
1001111101	511	10000101101	1023
1010000111	511	10000110101	93
1010010101	511	10001000111	341
1010011001	73	10001010011	341
1010100011	511	10001100011	341
1010100101	511	10001100101	1023
1010101111	511	10001101111	1023
1010110111	511	10010000001	1023
1010111101	511	10010001011	1023
1011001111	511	10010011001	341
1011010001	511	10010101001	33
1011011011	511	10010101111	341
1011110101	511	10011000101	1023
1011111001	511	10011001001	341
1100000001	73	10011010111	1023
1100010011	511	10011100111	1023
1100010101	511	10011101101	341
1100011111	511	10011110011	1023
1100100011	511	10011111111	1023
1100110001	511	10100001011	93
1100111011	511	10100001101	1023
1101001001	73	10100011001	1023
1101001111	511	10100011111	341
1101011011	511	10100100011	1023
1101100001	511	10100110001	1023
1101101011	511	10100111101	1023
1101101101	511	10101000011	1023
1101110011	511	10101010111	1023
1101111111	511	10101100001	93
1110000101	511	10101100111	341
1110001111	511	10101101011	1023
1110100001	73	10110000101	1023
1110110101	511	10110001111	1023
1110111001	511	10110010111	1023
1111000111	511	10110011011	341
1111001011	511	10110100001	1023
1111001101	511	10110101011	341
1111010101	511	10110111001	341
1111011001	511	10111000001	341
1111100011	511	10111000111	1023
1111101001	511	10111100101	1023
1111111011	511	10111110111	1023
		10111111011	1023

		$n = 11$	e
11000010011	1023		
11000010101	1023		
11000100011	33	100000000101	2047
11000100101	1023	100000010111	2047
11000110001	341	100000101011	2047
11000110111	1023	100000101101	2047
11001000011	1023	100001000111	2047
11001001111	1023	100001100011	2047
11001010001	341	100001100101	2047
11001011011	1023	100001110001	2047
11001111001	1023	100001111011	2047
11001111111	1023	100010001101	2047
11010000101	93	100010010101	2047
11010001001	1023	100010011111	2047
11010100111	93	100010101001	2047
11010101101	341	100010110001	2047
11010110101	1023	100011000011	89
11010111111	341	100011001111	2047
11011000001	1023	100011010001	2047
11011001101	341	100011100001	2047
11011010011	1023	100011100111	2047
11011011111	1023	100011101011	2047
11011110111	341	100011110101	2047
11011111101	1023	100100001101	2047
11100001111	341	100100010011	2047
11100010001	341	100100100101	2047
11100010111	1023	100100101001	2047
11100011101	1023	100100110111	89
11100100001	1023	100100111011	2047
11100101011	93	100100111101	2047
11100110101	341	100101000101	2047
11100111001	1023	100101001001	2047
11101000111	1023	100101010001	2047
11101001101	1023	100101011011	2047
11101010101	1023	100101110011	2047
11101011001	1023	100101110101	2047
11101100011	1023	100101111111	2047
11101111011	341	100110000011	2047
11101111101	1023	100110001111	2047
11110000001	341	100110101011	2047
11110000111	341	100110101101	2047
11110001101	1023	100110111001	2047
11110010011	1023	100111000111	2047
11110101001	341	100111011001	2047
11110110001	1023	100111100101	2047
11111000101	341	100111101111	89
11111011011	1023	100111110111	2047
11111101011	341	101000000001	2047
11111110011	1023	101000000111	2047
11111111001	1023	101000010011	2047
11111111111	11	101000010101	2047

IRREDUCIBLE POLYNOMIALS FOR THE MODULUS 2—*Concluded*

101000101001	2047	101110010101	2047	110100011101	2047	111001111101	2047
101001001001	2047	101110101111	2047	110100100111	2047	111010000001	2047
101001100001	2047	101110110111	2047	110100101101	2047	111010010011	2047
101001101101	2047	101110111101	2047	110101000001	2047	111010011111	2047
101001111001	2047	101111001001	2047	110101000111	2047	111010100011	2047
101001111111	2047	101111011011	2047	110101010101	2047	111010111011	2047
101010000101	2047	101111011101	2047	110101011001	2047	111011001001	89
101010010001	2047	101111100111	2047	110101100011	2047	111011001111	2047
101010011101	2047	101111101101	2047	110101101111	2047	111011011101	2047
101010100111	2047	110000001011	2047	110101110001	2047	111011110011	2047
101010101011	2047	110000001101	2047	110110010011	2047	111011111001	2047
101010110011	2047	110000011001	2047	110110011111	2047	111100001011	2047
101010110101	2047	110000011111	2047	110110101001	2047	111100011001	2047
101011010101	2047	110000110001	89	110110111011	2047	111100110001	2047
101011011111	2047	110001010111	2047	110110111101	2047	111100110111	2047
101011100011	23	110001100001	2047	110111001001	2047	111101011101	2047
101011101001	2047	110001101011	2047	110111010111	2047	111101101011	2047
101011101111	2047	110001110011	2047	110111011011	2047	111101101101	2047
101011110001	2047	110001110101	23	110111100001	2047	111101110101	2047
101011111011	2047	110010000101	2047	110111100111	2047	111101111001	89
101100000011	2047	110010001001	2047	110111110101	2047	111110000011	2047
101100001001	2047	110010010111	2047	110111111111	89	111110010001	2047
101100010001	2047	110010011011	2047	111000000101	2047	111110010111	2047
101100110011	2047	110010011101	2047	111000011101	2047	111110011011	2047
101100111111	2047	110010110011	2047	111000100001	2047	111110100111	2047
101101000001	2047	110010111111	2047	111000100111	2047	111110101101	2047
101101001011	2047	110011000111	2047	111000101011	2047	111110110101	2047
101101011001	2047	110011001101	2047	111000110011	2047	111111001101	2047
101101011111	2047	110011010011	2047	111000111001	2047	111111010011	2047
101101100101	2047	110011010101	2047	111001000111	2047	111111100101	2047
101101101111	2047	110011100011	2047	111001001011	2047	111111101001	2047
101101111101	2047	110011101001	2047	111001010101	2047	111111111011	89
101110000111	2047	110011110111	2047	111001011111	2047		
101110001011	2047	110100000011	2047	111001110001	2047		
101110010011	2047	110100001111	2047	111001111011	2047		

IRREDUCIBLE POLYNOMIALS FOR THE MODULUS 3

$n = 1$	e	1022	13	10111	40	12121	10	101201	242
		1102	13	10121	40	12212	80	101221	242
10		1112	13	10202	16			102101	242
11	2	1121	26	11002	80	$n = 5$	e	102112	121
12	1	1201	26	11021	20			102122	11
$n = 2$	e	1211	26	11101	40	100021	242	102202	121
		1222	13	11111	5	100022	121	102211	242
101	4			11122	80	100112	121	102221	22
112	8	$n = 4$	e	11222	80	100211	242	110002	121
122	8			12002	80	101011	242	110012	121
$n = 3$	e	10012	80	12011	20	101012	121	110021	242
		10022	80	12101	40	101102	121	110101	242
1021	26	10102	16	12112	80	101122	121	110111	242

IRREDUCIBLE POLYNOMIALS FOR THE MODULUS 2—*Continued*

		$n = 10$	e
1001001011	73		
1001011001	511		
1001011111	511	10000001001	1023
1001100101	73	10000001111	341
1001101001	511	10000011011	1023
1001101111	511	10000011101	341
1001110111	511	10000100111	1023
1001111101	511	10000101101	1023
1010000111	511	10000110101	93
1010010101	511	10001000111	341
1010011001	73	10001010011	341
1010100011	511	10001100011	341
1010100101	511	10001100101	1023
1010101111	511	10001101111	1023
1010110111	511	10010000001	1023
1010111101	511	10010001011	1023
1011001111	511	10010011001	341
1011010001	511	10010101001	33
1011011011	511	10010101111	341
1011110101	511	10011000101	1023
1011111001	511	10011001001	341
1100000001	73	10011010111	1023
1100010011	511	10011100111	1023
1100010101	511	10011101101	341
1100011111	511	10011110011	1023
1100100011	511	10011111111	1023
1100110001	511	10100001011	93
1100111011	511	10100001101	1023
1101001001	73	10100011001	1023
1101001111	511	10100011111	341
1101011011	511	10100100011	1023
1101100001	511	10100110001	1023
1101101011	511	10100111101	1023
1101101101	511	10101000011	1023
1101110011	511	10101010111	1023
1101111111	511	10101100001	93
1110000101	511	10101100111	341
1110001111	511	10101101011	1023
1110100001	73	10110000101	1023
1110110101	511	10110001111	1023
1110111001	511	10110010111	1023
1111000111	511	10110011011	341
1111001011	511	10110100001	1023
1111001101	511	10110101011	341
1111010101	511	10110111001	341
1111011001	511	10111000001	341
1111100011	511	10111000111	1023
1111101001	511	10111100101	1023
1111111011	511	10111110111	1023
		10111111011	1023

		$n = 11$	e
11000010011	1023		
11000010101	1023		
11000100011	33	100000000101	2047
11000100101	1023	100000010111	2047
11000110001	341	100000101011	2047
11000110111	1023	100000101101	2047
11001000011	1023	100001000111	2047
11001001111	1023	100001100011	2047
11001010001	341	100001100101	2047
11001011011	1023	100001110001	2047
11001111001	1023	100001111011	2047
11001111111	1023	100010001101	2047
11010000101	93	100010010101	2047
11010001001	1023	100010011111	2047
11010100111	93	100010101001	2047
11010101101	341	100010110001	2047
11010110101	1023	100011000011	89
11010111111	341	100011001111	2047
11011000001	1023	100011010001	2047
11011001101	341	100011100001	2047
11011010011	1023	100011100111	2047
11011011111	1023	100011101011	2047
11011110111	341	100011110101	2047
11011111101	1023	100100001101	2047
11100001111	341	100100010011	2047
11100010001	341	100100100101	2047
11100010111	1023	100100101001	2047
11100011101	1023	100100110111	89
11100100001	1023	100100111011	2047
11100101011	93	100100111101	2047
11100110101	341	100101000101	2047
11100111001	1023	100101001001	2047
11101000111	1023	100101010001	2047
11101001101	1023	100101011011	2047
11101010101	1023	100101110011	2047
11101011001	1023	100101110101	2047
11101100011	1023	100101111111	2047
11101111011	341	100110000011	2047
11101111101	1023	100110001111	2047
11110000001	341	100110101011	2047
11110000111	341	100110101101	2047
11110001101	1023	100110111001	2047
11110010011	1023	100111000111	2047
11110101001	341	100111011001	2047
11110110001	1023	100111100101	2047
11111000101	341	100111101111	89
11111011011	1023	100111110111	2047
11111101011	341	101000000001	2047
11111110011	1023	101000000111	2047
11111111001	1023	101000010011	2047
11111111111	11	101000010101	2047

IRREDUCIBLE POLYNOMIALS FOR THE MODULUS 2—*Concluded*

101000101001	2047	101110010101	2047	110100011101	2047	111001111101	2047
101001001001	2047	101110101111	2047	110100100111	2047	111010000001	2047
101001100001	2047	101110110111	2047	110100101101	2047	111010010011	2047
101001101101	2047	101110111101	2047	110101000001	2047	111010011111	2047
101001111001	2047	101111001001	2047	110101000111	2047	111010100011	2047
101001111111	2047	101111011011	2047	110101010101	2047	111010111011	2047
101010000101	2047	101111011101	2047	110101011001	2047	111011001001	89
101010010001	2047	101111100111	2047	110101100011	2047	111011001111	2047
101010011101	2047	101111101101	2047	110101101111	2047	111011011101	2047
101010100111	2047	110000001011	2047	110101110001	2047	111011110011	2047
101010101011	2047	110000001101	2047	110110010011	2047	111011111001	2047
101010110011	2047	110000011001	2047	110110011111	2047	111100001011	2047
101010110101	2047	110000011111	2047	110110101001	2047	111100011001	2047
101011010101	2047	110000110001	89	110110111011	2047	111100110001	2047
101011011111	2047	110001010111	2047	110110111101	2047	111100110111	2047
101011100011	23	110001100001	2047	110111001001	2047	111101011101	2047
101011101001	2047	110001101011	2047	110111010111	2047	111101101011	2047
101011101111	2047	110001110011	2047	110111011011	2047	111101101101	2047
101011110001	2047	110001110101	23	110111100001	2047	111101110101	2047
101011111011	2047	110010000101	2047	110111100111	2047	111101111001	89
101100000011	2047	110010001001	2047	110111110101	2047	111110000011	2047
101100001001	2047	110010010111	2047	110111111111	89	111110010001	2047
101100010001	2047	110010011011	2047	111000000101	2047	111110010111	2047
101100110011	2047	110010011101	2047	111000011101	2047	111110011011	2047
101100111111	2047	110010110011	2047	111000100001	2047	111110100111	2047
101101000001	2047	110010111111	2047	111000100111	2047	111110101101	2047
101101001011	2047	110011000111	2047	111000101011	2047	111110110101	2047
101101011001	2047	110011001101	2047	111000110011	2047	111111001101	2047
101101011111	2047	110011010011	2047	111000111001	2047	111111010011	2047
101101100101	2047	110011010101	2047	111001000111	2047	111111100101	2047
101101101111	2047	110011100011	2047	111001001011	2047	111111101001	2047
101101111101	2047	110011101001	2047	111001010101	2047	111111111011	89
101110000111	2047	110011110111	2047	111001011111	2047		
101110001011	2047	110100000011	2047	111001110001	2047		
101110010011	2047	110100001111	2047	111001111011	2047		

IRREDUCIBLE POLYNOMIALS FOR THE MODULUS 3

$n = 1$	e	1022	13	10111	40	12121	10	101201	242
		1102	13	10121	40	12212	80	101221	242
10		1112	13	10202	16			102101	242
11	2	1121	26	11002	80	$n = 5$	e	102112	121
12	1	1201	26	11021	20			102122	11
$n = 2$	e	1211	26	11101	40	100021	242	102202	121
		1222	13	11111	5	100022	121	102211	242
101	4			11122	80	100112	121	102221	22
112	8	$n = 4$	e	11222	80	100211	242	110002	121
122	8			12002	80	101011	242	110012	121
$n = 3$	e	10012	80	12011	20	101012	121	110021	242
		10022	80	12101	40	101102	121	110101	242
1021	26	10102	16	12112	80	101122	121	110111	242

IRREDUCIBLE POLYNOMIALS FOR THE MODULUS 3—*Continued*

110122	121	1012001	182	1122002	104	10001012	1093	10120021	2186
111011	242	1012012	728	1122122	104	10001102	1093	10120112	1093
111121	242	1012021	364	1122202	728	10001111	2186	10120202	1093
111211	242	1012112	728	1122221	364	10001201	2186	10121002	1093
111212	121	1020001	52	1200002	728	10001212	1093	10121102	1093
112001	242	1020101	52	1200022	56	10002112	1093	10121201	2186
112022	121	1020112	728	1200121	364	10002122	1093	10121222	1093
112102	11	1020122	728	1201001	364	10002211	2186	10122001	2186
112111	242	1021021	364	1201111	182	10002221	2186	10122011	2186
112201	242	1021102	56	1201121	182	10010122	1093	10122022	1093
112202	121	1021112	728	1201201	364	10010222	1093	10122212	1093
120001	242	1021121	91	1201202	728	10011002	1093	10122221	2186
120011	242	1022011	364	1202002	728	10011101	2186	10200001	2186
120022	121	1022102	56	1202021	28	10011211	2186	10200002	1093
120202	121	1022111	182	1202101	364	10012001	2186	10200101	2186
120212	121	1022122	728	1202122	728	10012022	1093	10200112	1093
120221	242	1100002	728	1202222	728	10012111	2186	10200202	1093
121012	121	1100012	56	1210001	364	10012202	1093	10200211	2186
121111	242	1100111	364	1210021	364	10020121	2186	10201021	2186
121112	121	1101002	728	1210112	728	10020221	2186	10201022	1093
121222	121	1101011	28	1210202	728	10021001	2186	10201121	2186
122002	121	1101101	364	1210211	91	10021112	1093	10201222	1093
122021	242	1101112	728	1211021	182	10021202	1093	10202011	2186
122101	242	1101212	728	1211201	91	10022002	1093	10202012	1093
122102	121	1102001	364	1211212	728	10022021	2186	10210001	2186
122201	22	1102111	91	1212011	91	10022101	2186	10210121	2186
122212	121	1102121	91	1212022	728	10022212	1093	10210202	1093
		1102201	364	1212121	14	10100011	2186	10211101	2186
$n = 6$	e	1102202	728	1212122	728	10100012	1093	10211111	2186
1000012	728	1110001	364	1212212	728	10100102	1093	10211122	1093
1000022	728	1110011	364	1220102	728	10100122	1093	10211221	2186
1000111	364	1110122	728	1220111	182	10100201	2186	10212011	2186
1000121	364	1110202	728	1220212	728	10100221	2186	10212022	1093
1000201	52	1110221	182	1221001	182	10101101	2186	10212101	2186
1001012	728	1111012	728	1221002	104	10101112	1093	10212112	1093
1001021	364	1111021	182	1221112	104	10101202	1093	10212212	1093
1001101	91	1111111	7	1221202	728	10101211	2186	10220002	1093
1001122	104	1111112	728	1221211	364	10102102	1093	10220101	2186
1001221	182	1111222	728	1222022	728	10102201	2186	10220222	1093
1002011	364	1112011	91	1222102	728	10110022	1093	10221122	1093
1002022	728	1112201	182	1222112	104	10110101	2186	10221202	1093
1002101	182	1112222	728	1222211	364	10110211	2186	10221212	1093
1002112	104	1120102	728	1222222	728	10111001	2186	10221221	2186
1002211	91	1120121	91			10111102	1093	10222012	1093
1010201	52	1120222	728	$n = 7$	e	10111121	2186	10222021	2186
1010212	728	1121012	728			10111201	2186	10222111	2186
1010222	728	1121102	728	10000102	1093	10112002	1093	10222202	1093
1011001	91	1121122	104	10000121	2186	10112012	1093	10222211	2186
1011011	364	1121212	728	10000201	2186	10112021	2186	11000101	2186
1011022	728	1121221	364	10000222	1093	10112111	2186	11000222	1093
1011122	728	1122001	91	10001011	2186	10112122	1093	11001022	1093

IRREDUCIBLE POLYNOMIALS FOR THE MODULUS 3—*Concluded*

11001112	1093	11110001	2186	11211022	1093	12012221	2186	12121022	1093
11001211	2186	11110012	1093	11211122	1093	12020002	1093	12121102	1093
11002012	1093	11110111	2186	11211212	1093	12020021	2186	12121121	2186
11002022	1093	11110112	1093	11211221	2186	12020122	1093	12122012	1093
11002121	2186	11110211	2186	11212012	1093	12020222	1093	12122122	1093
11002202	1093	11110222	1093	11212112	1093	12021101	2186	12200101	2186
11010001	2186	11111011	2186	11212202	1093	12021212	1093	12200102	1093
11010022	1093	11111021	2186	11220001	2186	12022001	2186	12201011	2186
11010121	2186	11111201	2186	11220112	1093	12022111	2186	12201022	1093
11010221	2186	11111222	1093	11220211	2186	12022201	2186	12201121	2186
11011111	2186	11112011	2186	11221022	1093	12100001	2186	12201122	1093
11011202	1093	11112221	2186	11221102	1093	12100021	2186	12201202	1093
11012002	1093	11120102	1093	11221112	1093	12100111	2186	12201212	1093
11012102	1093	11120111	2186	11221121	2186	12100222	1093	12202001	2186
11012212	1093	11120122	1093	11222011	2186	12101011	2186	12202111	2186
11020021	2186	11120212	1093	11222102	1093	12101021	2186	12202112	1093
11020022	1093	11120221	2186	11222122	1093	12101201	2186	12202222	1093
11020102	1093	11121001	2186	11222201	2186	12101212	1093	12210002	1093
11020112	1093	11121101	2186	11222221	2186	12101222	1093	12210112	1093
11020201	2186	11121202	1093	12000121	2186	12102001	2186	12210211	2186
11020222	1093	11122021	2186	12000202	1093	12102121	2186	12211021	2186
11021111	2186	11122112	1093	12001021	2186	12102212	1093	12211201	2186
11021122	1093	11122201	2186	12001112	1093	12110111	2186	12211211	2186
11021201	2186	11122222	1093	12001211	2186	12110122	1093	12211222	1093
11021212	1093	11200201	2136	12002011	2186	12110201	2186	12212012	1093
11022101	2186	11200202	1093	12002021	2186	12110212	1093	12212102	1093
11022122	1093	11201012	1093	12002101	2186	12110221	2186	12212122	1093
11022211	2186	11201021	2186	12002222	1093	12111002	1093	12212201	2186
11022221	2186	11201101	2186	12010021	2186	12111101	2186	12212221	2186
11100002	1093	11201111	2186	12010022	1093	12111202	1093	12220001	2186
11100022	1093	11201221	2186	12010102	1093	12112022	1093	12220012	1093
11100121	2186	11201222	1093	12010121	2186	12112102	1093	12220022	1093
11100212	1093	11202002	1093	12010201	2186	12112121	2186	12220202	1093
11101012	1093	11202121	2186	12010211	2186	12112211	2186	12221002	1093
11101022	1093	11202211	2186	12011102	1093	12120002	1093	12221021	2186
11101102	1093	11202212	1093	12011111	2186	12120011	2186	12221111	2186
11101111	2186	11210002	1093	12011212	1093	12120112	1093	12221122	1093
11101121	2186	11210011	2186	12011221	2186	12120121	2186	12221221	2186
11102002	1093	11210021	2186	12012112	1093	12120211	2186	12222011	2186
11102111	2186	11210101	2186	12012122	1093	12120212	1093	12222101	2186
11102222	1093	11211001	2186	12012202	1093	12121012	1093	12222211	2186

IRREDUCIBLE POLYNOMIALS FOR THE MODULUS 5

$n = 1$	e	$n = 2$	e	124	12	$n = 3$	e	1033	124	1131	62
				133	24			1042	124	1134	31
10		102	8	134	12	1011	62	1043	124	1141	62
11	2	103	8	141	6	1014	31	1101	62	1143	124
12	4	111	3	142	24	1021	62	1102	124	1201	62
13	4	112	24			1024	31	1113	124	1203	124
14	1	123	24			1032	124	1114	31	1213	124

IRREDUCIBLE POLYNOMIALS FOR THE MODULUS 5—*Continued*

1214	31	10431	156	12311	39	14123	208	101033	284	103022	3124
1222	124	10442	624	12312	208	14134	104	101103	3124	103023	3124
1223	124	10443	624	12324	312	14143	624	101104	781	103101	1562
1242	124	11004	312	12332	624	14144	312	101141	1562	103104	781
1244	31	11013	624	12333	208	14202	624	101142	3124	103111	1562
1302	124	11023	624	12344	104	14214	312	101203	3124	103112	3124
1304	31	11024	104	12401	156	14224	104	101204	781	103143	3124
1311	62	11032	624	12414	104	14231	156	101212	3124	103144	71
1312	124	11041	52	12422	208	14232	208	101213	284	103211	1562
1322	124	11042	624	12433	624	14242	624	101301	1562	103212	3124
1323	124	11101	78	12434	312	14243	208	101302	3124	103221	1562
1341	62	11113	624	12443	208	14301	156	101312	284	103223	3124
1343	124	11114	312	13004	312	14303	624	101313	3124	103232	3124
1403	124	11124	104	13012	624	14312	624	101401	1562	103233	3124
1404	31	11133	208	13023	624	14314	312	101402	3124	103313	3124
1411	62	11142	208	13031	13	14331	78	101443	3124	103314	781
1412	124	11202	624	13032	624	14402	208	101444	781	103322	3124
1431	62	11212	624	13043	624	14411	52	102001	1562	103324	781
1434	31	11213	208	13044	104	14413	624	102004	781	103332	3124
1442	124	11221	156	13102	208	14441	26	102012	3124	103333	3124
1444	31	11222	208	13121	52	14444	312	102013	3124	103401	1562
		11234	104	13124	312			102021	1562	103404	781
$n = 4$	e	11244	312	13131	26	$n = 5$	e	102024	781	103413	3124
		11301	156	13133	624			102112	3124	103414	781
10002	16	11303	624	13201	78	100041	1562	102114	781	103441	142
10003	16	11321	39	13203	624	100042	3124	102121	1562	103442	3124
10014	312	11342	624	13232	624	100043	3124	102122	3124	104021	1562
10024	312	11344	312	13234	312	100044	781	102131	1562	104024	781
10034	312	11402	208	13241	156	100102	3124	102134	781	104031	142
10044	312	11411	13	13302	624	100114	781	102202	3124	104034	71
10102	48	11414	312	13314	104	100124	71	102203	3124	104101	1562
10111	78	11441	52	13322	624	100132	3124	102211	1562	104103	3124
10122	624	11443	624	13323	208	100143	3124	102213	3124	104111	142
10123	624	12004	312	13334	312	100201	1562	102242	284	104114	781
10132	624	12013	624	13341	78	100212	3124	102244	781	104202	3124
10133	624	12014	104	13342	208	100222	284	102302	3124	104204	781
10141	39	12021	26	13401	156	100231	1562	102303	3124	104241	1562
10203	48	12022	624	13413	208	100244	781	102312	3124	104243	3124
10221	39	12033	624	13423	624	100304	781	102314	781	104301	1562
10223	208	12042	624	13424	312	100313	3124	102341	1562	104303	3124
10231	78	12102	208	13432	208	100323	284	102343	284	104342	3124
10233	208	12121	13	13444	104	100334	781	102411	1562	104344	781
10303	48	12123	624	14004	312	100341	1562	102413	3124	104402	3124
10311	156	12131	52	14011	52	100403	3124	102423	3124	104404	781
10313	208	12134	312	14012	624	100411	1562	102424	781	104411	1562
10341	156	12201	39	14022	624	100421	142	102431	1562	104414	71
10343	208	12203	624	14033	624	100433	3124	102434	781	110004	781
10402	48	12211	156	14034	104	100442	3124	103002	3124	110014	781
10412	624	12222	624	14043	624	101022	3124	103003	3124	110041	1562
10413	624	12224	312	14101	39	101023	3124	103011	1562	110123	3124
10421	156	12302	624	14112	208	101032	284	103014	781	110131	1562

IRREDUCIBLE POLYNOMIALS FOR THE MODULUS 5—*Continued*

110142	3124	112133	3124	114102	3124	121103	284	123142	3124	130233	3124
110144	781	112142	3124	114132	3124	121131	1562	123224	781	130241	1562
110202	284	112143	284	114141	1562	121144	781	123231	1562	130242	3124
110213	3124	112201	1562	114201	1562	121201	1562	123242	3124	130304	781
110232	3124	112212	3124	114204	71	121202	3124	123303	3124	130313	3124
110243	3124	112214	781	114233	3124	121223	3124	123311	1562	130323	3124
110244	781	112234	781	114242	3124	121232	44	123331	1562	130331	1562
110301	1562	112241	1562	114314	781	121233	3124	123341	142	130341	1562
110303	3124	112243	3124	114321	1562	121244	781	123344	781	130342	3124
110322	3124	112301	1562	114322	3124	121304	781	123402	3124	130343	3124
110331	1562	112311	1562	114331	1562	121334	781	123411	1562	130401	142
110333	3124	112313	3124	114343	3124	121342	3124	123412	3124	130414	781
110343	3124	112314	781	114401	1562	121413	3124	123413	3124	130431	1562
110403	3124	112323	3124	114403	3124	121422	3124	123421	1562	130442	3124
110411	1562	112334	71	114424	781	121424	781	123433	284	130444	781
110421	1562	112342	3124	114431	22	121432	3124	123444	781	131003	3124
110432	3124	112422	3124	114434	781	121441	1562	124001	142	131011	1562
110441	1562	112433	3124	114442	3124	122003	3124	124011	1562	131012	3124
110442	3124	112441	1562	120003	3124	122004	781	124022	3124	131013	3124
110444	781	113002	3124	120013	3124	122033	3124	124023	3124	131022	3124
111003	284	113004	781	120042	3124	122043	3124	124024	781	131034	781
111013	3124	113034	781	120104	71	122112	3124	124034	781	131042	3124
111021	1562	113044	781	120111	1562	122123	284	124043	3124	131112	3124
111022	3124	113103	3124	120134	781	122124	781	124114	11	131121	1562
111024	781	113111	1562	120141	1562	122132	3124	124123	3124	131123	3124
111032	3124	113134	781	120143	3124	122141	142	124132	3124	131133	3124
111044	781	113142	284	120201	1562	122142	3124	124133	3124	131144	781
111102	3124	113143	3124	120212	3124	122214	781	124202	284	131201	1562
111114	781	113211	1562	120222	3124	122224	781	124203	3124	131231	1562
111123	3124	113222	3124	120234	781	122233	3124	124221	1562	131243	3124
111212	44	113224	71	120242	3124	122301	1562	124231	1562	131303	3124
111224	781	113231	1562	120243	3124	122312	3124	124232	3124	131304	781
111231	1562	113241	1562	120244	781	122333	3124	124244	781	131322	3124
111234	781	113243	284	120321	1562	122341	1562	124304	781	131332	3124
111301	1562	113304	781	120332	3124	122344	71	124313	3124	131333	44
111311	1562	113312	3124	120343	3124	122403	3124	124321	1562	131341	1562
111312	3124	113321	1562	120344	781	122414	781	124402	3124	131402	284
111324	71	113323	3124	120401	1562	122421	1562	124412	3124	131403	3124
111334	781	113324	781	120402	3124	122422	3124	124414	781	131434	781
111401	142	113332	3124	120424	781	122423	3124	124423	284	131441	1562
111404	781	113342	3124	120431	1562	122434	781	124433	3124	132001	1562
111423	3124	113412	3124	120432	3124	122444	781	130002	3124	132002	3124
111431	1562	113422	3124	120441	1562	123014	781	130012	3124	132032	3124
111433	3124	113434	781	121002	3124	123021	1562	130043	3124	132042	3124
111442	3124	114001	1562	121012	3124	123033	3124	130103	3124	132102	3124
112012	3124	114011	1562	121013	3124	123034	781	130104	781	132111	1562
112023	3124	114012	3124	121014	781	123102	3124	130121	1562	132122	3124
112032	3124	114014	781	121023	3124	123113	3124	130133	3124	132123	3124
112034	781	114024	781	121031	1562	123114	781	130134	781	132124	781
112104	781	114033	3124	121043	3124	123133	3124	130144	781	132131	1562
112113	3124	114044	781	121102	3124	123141	1562	130224	781	132141	1562

IRREDUCIBLE POLYNOMIALS FOR THE MODULUS 5—*Concluded*

132204	781	133343	3124	140001	1562	141101	1562	142304	781	143414	781
132213	3124	133403	3124	140011	1562	141104	71	142311	1562	143431	1562
132232	3124	133411	1562	140044	781	141122	3124	142313	3124	143442	3124
132241	142	133412	3124	140102	3124	141132	3124	142331	1562	143443	284
132244	781	133432	3124	140114	781	141134	781	142342	3124	144004	781
132311	1562	133443	3124	140124	781	141143	3124	142344	781	144011	1562
132321	1562	133444	781	140133	3124	141204	781	142401	1562	144013	3124
132332	3124	134004	71	140141	1562	141213	3124	142412	3124	144014	781
132413	3124	134014	781	140143	3124	141214	781	142432	3124	144021	1562
132421	1562	134021	1562	140144	781	141221	142	142442	284	144032	3124
132422	284	134022	3124	140202	3124	141231	1562	142443	3124	144041	1562
132433	3124	134023	3124	140204	781	141313	44	143001	1562	144102	3124
132443	3124	134031	1562	140223	3124	141321	1562	143003	3124	144104	781
132444	71	134042	3124	140232	3124	141331	1562	143031	1562	144121	1562
133011	1562	134103	3124	140234	781	141334	781	143041	1562	144131	1562
133024	781	134111	1562	140242	3124	141403	3124	143113	3124	144134	11
133031	1562	134113	3124	140303	284	141411	1562	143123	3124	144143	3124
133032	3124	134122	284	140312	3124	141422	3124	143131	1562	144211	1562
133103	3124	134132	3124	140333	3124	142013	3124	143201	1562	144223	3124
133112	3124	134201	1562	140341	1562	142022	3124	143213	3124	144224	781
133113	3124	134212	3124	140342	3124	142031	1562	143221	1562	144234	781
133114	781	134224	781	140422	3124	142033	3124	143222	3124	144242	3124
133124	781	134302	3124	140434	781	142123	3124	143224	781	144301	142
133132	284	134303	284	140441	1562	142132	3124	143233	3124	144304	781
133141	1562	134324	781	140443	3124	142144	781	143243	3124	144332	3124
133202	3124	134333	3124	141002	284	142204	781	143314	781	144343	3124
133214	781	134334	781	141012	3124	142211	1562	143321	142	144403	3124
133234	781	134341	1562	141021	1562	142212	3124	143323	3124	144433	3124
133241	1562	134411	22	141023	3124	142214	781	143334	781	144444	781
133244	71	134422	3124	141024	781	142222	3124	143342	284		
133321	1562	134432	3124	141033	3124	142231	142	143344	781		
133334	781	134433	3124	141041	1562	142243	3124	143402	3124		

IRREDUCIBLE POLYNOMIALS FOR THE MODULUS 7

$n = 1$	e	116	16	$n = 3$	e	1062	342	1163	171	1263	171
		122	24			1065	171	1165	171	1264	342
10		123	48	1002	18	1101	114	1201	38	1304	342
11	2	125	48	1003	9	1103	171	1203	171	1306	57
12	6	131	8	1004	18	1112	342	1214	342	1311	38
13	3	135	48	1005	9	1115	171	1216	57	1314	342
14	6	136	16	1011	114	1124	342	1223	171	1322	342
15	3	141	8	1016	57	1126	57	1226	57	1325	171
16	1	145	48	1021	38	1131	38	1233	171	1333	171
$n = 2$	e	146	16	1026	19	1135	171	1235	171	1334	342
		152	24	1032	342	1143	171	1242	342	1335	171
101	4	153	48	1035	171	1146	57	1245	171	1336	19
102	12	155	48	1041	114	1151	114	1251	114	1341	38
104	12	163	48	1046	57	1152	342	1255	171	1343	171
113	48	164	24	1052	342	1153	171	1261	114	1352	342
114	24	166	16	1055	171	1154	342	1262	342	1354	342

IRREDUCIBLE POLYNOMIALS FOR THE MODULUS 7—*Continued*

		$n = 4$	e								
1362	342	10011	400	10414	600	11134	240	11556	800	12303	2400
1366	57	10012	1200	10422	600	11136	800	11562	1200	12304	240
1401	114	10014	1200	10433	2400	11141	400	11566	160	12311	200
1403	171	10023	480	10443	2400	11152	1200	11602	240	12323	2400
1413	171	10025	480	10452	600	11153	2400	11605	2400	12325	2400
1416	19	10026	160	10462	1200	11161	100	11614	600	12332	120
1422	342	10053	480	10464	600	11163	480	11625	2400	12345	480
1425	171	10055	480	10503	96	11166	800	11626	800	12346	800
1431	38	10056	160	10505	96	11201	200	11631	40	12351	100
1432	342	10061	400	10515	2400	11204	600	11643	2400	12354	600
1433	171	10062	1200	10524	1200	11213	2400	11646	160	12356	800
1434	342	10064	1200	10525	2400	11223	2400	11652	600	12361	200
1444	342	10103	96	10531	80	11225	2400	11653	2400	12363	2400
1446	19	10106	32	10533	2400	11232	60	11654	300	12365	2400
1453	171	10111	400	10536	800	11233	2400	11664	600	12402	1200
1455	171	10112	600	10541	80	11236	800	11665	2400	12403	2400
1461	114	10121	200	10543	2400	11241	400	11666	800	12406	800
1465	171	10135	2400	10546	800	11244	1200	12002	1200	12412	15
1504	342	10145	2400	10554	1200	11245	2400	12006	160	12414	1200
1506	19	10151	200	10555	2400	11252	240	12016	800	12421	25
1511	114	10161	400	10565	2400	11254	300	12025	2400	12431	80
1513	171	10162	600	10603	96	11266	160	12032	1200	12435	2400
1521	114	10203	96	10606	32	11321	200	12044	75	12442	1200
1524	342	10205	96	10613	2400	11323	2400	12051	100	12454	1200
1532	342	10211	200	10621	400	11324	150	12055	2400	12456	800
1534	342	10214	1200	10623	2400	11331	400	12064	1200	12462	300
1542	342	10224	600	10632	240	11332	300	12066	800	12465	2400
1545	171	10236	800	10635	2400	11334	600	12101	200	12466	160
1552	342	10246	800	10636	800	11351	200	12102	600	12521	50
1556	57	10254	600	10642	240	11355	2400	12116	800	12522	600
1563	171	10261	200	10645	2400	11356	160	12123	2400	12526	800
1564	342	10264	1200	10646	800	11362	120	12126	800	12531	200
1565	171	10305	96	10651	400	11364	1200	12134	60	12532	1200
1566	57	10306	32	10653	2400	11365	2400	12135	2400	12534	300
1604	342	10316	800	10663	2400	11405	2400	12136	800	12552	600
1606	57	10322	1200	11001	400	11406	800	12141	400	12553	2400
1612	342	10326	800	11003	480	11412	1200	12142	1200	12555	480
1615	171	10333	2400	11013	2400	11415	480	12143	2400	12561	400
1621	114	10334	240	11026	800	11422	1200	12151	100	12563	2400
1623	171	10335	2400	11031	400	11423	2400	12154	240	12564	120
1632	342	10343	2400	11042	75	11434	1200	12165	480	12601	400
1636	19	10344	240	11054	300	11443	2400	12203	2400	12612	150
1641	114	10345	2400	11056	800	11455	2400	12205	2400	12626	800
1644	342	10352	1200	11062	1200	11463	2400	12213	480	12636	800
1653	171	10356	800	11063	2400	11504	1200	12214	1200	12643	2400
1654	342	10366	800	11101	400	11511	50	12224	1200	12644	75
1655	171	10405	96	11103	2400	11523	2400	12226	800	12652	1200
1656	57	10406	32	11105	2400	11533	2400	12231	400	12655	2400
1662	342	10412	1200	11111	5	11542	75	12246	800	12664	1200
1664	342			11112	1200	11545	2400	12253	2400	12665	480
				11124	75	11551	400	12266	800	13004	1200

IRREDUCIBLE POLYNOMIALS FOR THE MODULUS 7—*Continued*

13005	480	13441	20	14165	2400	14563	2400	15313	2400	16024	300
13011	400	13443	2400	14204	1200	14566	800	15315	2400	16026	800
13015	2400	13445	2400	14205	2400	14622	150	15321	100	16032	150
13022	300	13455	2400	14206	800	14624	600	15324	600	16041	400
13023	2400	13456	800	14211	400	14625	2400	15326	800	16056	800
13031	50	13465	2400	14214	15	14631	100	15335	480	16063	2400
13044	1200	13501	80	14222	75	14632	600	15336	800	16101	400
13053	2400	13506	800	14232	240	14634	1200	15342	120	16103	2400
13065	2400	13512	600	14233	2400	14653	480	15353	2400	16105	2400
13103	2400	13513	2400	14244	1200	14654	600	15355	2400	16111	100
13106	800	13516	800	14251	400	14656	800	15361	200	16113	480
13115	2400	13521	200	14255	2400	14661	40	15402	1200	16116	800
13126	800	13522	300	14263	2400	14662	1200	15403	2400	16122	1200
13135	2400	13525	2400	14264	300	14666	800	15406	800	16123	2400
13142	1200	13533	480	14265	480	15002	1200	15412	300	16131	400
13151	400	13535	2400	14302	1200	15006	160	15415	2400	16144	240
13155	2400	13544	120	14314	150	15014	1200	15416	160	16146	800
13161	400	13553	2400	14325	2400	15016	800	15424	1200	16154	150
13166	160	13556	800	14335	2400	15021	100	15426	800	16161	10
13204	1200	13562	600	14341	25	15025	2400	15432	1200	16162	1200
13205	2400	13611	40	14346	800	15034	150	15441	80	16201	200
13206	800	13612	1200	14353	2400	15042	1200	15445	2400	16204	600
13213	2400	13616	800	14354	1200	15055	2400	15451	50	16216	160
13214	300	13623	480	14361	400	15066	800	15462	30	16222	240
13215	480	13624	600	14363	480	15101	200	15464	1200	16224	300
13221	400	13626	800	14402	600	15102	600	15511	400	16231	400
13225	2400	13641	100	14404	600	15115	480	15513	2400	16234	1200
13234	1200	13642	600	14415	2400	15121	100	15514	120	16235	2400
13242	240	13644	1200	14425	2400	15124	240	15522	600	16242	60
13243	2400	13652	75	14426	800	15131	400	15523	2400	16243	2400
13252	150	13654	600	14431	20	15132	1200	15525	480	16246	800
13261	400	13655	2400	14433	2400	15133	2400	15541	200	16253	2400
13264	30	14004	1200	14435	2400	15144	60	15542	1200	16255	2400
13302	1200	14005	480	14442	1200	15145	2400	15544	300	16263	2400
13311	400	14015	2400	14444	1200	15146	800	15551	25	16312	120
13313	480	14023	2400	14446	800	15153	2400	15552	600	16314	1200
13323	2400	14034	1200	14451	80	15156	800	15556	800	16315	2400
13324	1200	14041	25	14452	300	15166	800	15601	400	16321	200
13331	50	14052	300	14463	480	15203	2400	15614	1200	16325	2400
13336	800	14053	2400	14501	80	15205	2400	15615	480	16326	160
13345	2400	14061	400	14506	800	15216	800	15622	1200	16341	400
13355	2400	14065	2400	14512	600	15223	2400	15625	2400	16342	300
13364	75	14103	2400	14523	2400	15236	800	15633	2400	16344	600
13402	600	14106	800	14526	800	15241	400	15634	150	16351	200
13404	600	14111	400	14534	120	15254	1200	15646	800	16353	2400
13413	480	14116	160	14543	480	15256	800	15656	800	16354	75
13421	80	14121	400	14545	2400	15263	480	15662	75	16405	2400
13422	300	14125	2400	14551	200	15264	1200	16001	400	16406	800
13432	1200	14132	1200	14552	300	15303	2400	16003	480	16413	2400
13434	1200	14145	2400	14555	2400	15304	240	16012	1200	16425	2400
13436	800	14156	800	14562	600	15311	200	16013	2400	16433	2400

IRREDUCIBLE POLYNOMIALS FOR THE MODULUS 7—*Concluded*

16444	1200	16504	1200	16532	150	16602	240	16622	600	16641	40
16452	1200	16512	1200	16535	2400	16605	2400	16623	2400	16655	2400
16453	2400	16516	160	16543	2400	16614	600	16624	300	16656	800
16462	1200	16521	400	16553	2400	16615	2400	16633	2400	16664	600
16465	480	16526	800	16561	25	16616	800	16636	160		

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THE FUNDAMENTAL GROUP OF A GROUP MANIFOLD

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The study of the interrelations between group structures and topological structures is one of the interesting recent developments of group theory. It has become well known that a space $\mathfrak{G}$ which is at the same time a continuous group cannot be of arbitrary topological structure. Schreier,[1] for example, proved that the so-called fundamental group G of $\mathfrak{G}$ must be abelian. Our purpose is to show further that the "size" of G is limited by the dimension of $\mathfrak{G}$, that is, by the "number of parameters." We shall, in fact, prove that *the maximum number of linearly independent elements of G cannot exceed the dimension of $\mathfrak{G}$.*

The proof depends on certain results of Hopf and of Alexandroff, applied to a preliminary construction—essentially simple although involving a number of details—by means of which the covering manifold of $\mathfrak{G}$ is made compact by the addition of specially chosen ideal elements.

I. Preliminary Construction

1. The topology of a complex is completely determined by its defining relations. This is obvious for finite complexes but for infinite complexes depends, strictly speaking, on additional conventions. We shall agree to consider each infinite simplicial complex K as immersed in a Hilbert parallelotope: $0 \leqq x_i \leqq 1/i$ $(i = 1, 2, \cdots)$, and this in such a way that distinct vertices lie in distinct coördinate axes. Assuming that at most a finite number of cells of K are incident to any one cell of K, the particular property which we desire is now determined: if $p^1, p^2, \cdots$ are points contained respectively in the distinct cells $e^1, e^2, \cdots$ of K, then the set $\{p^1, p^2, \cdots\}$ has no limit point in K. For evidently $p^i \to 0$, where 0 is the origin, and 0 is contained in no cell of K. *Corollary:* under the same incidence hypothesis, a closed compact subset of K will meet at most a finite number of cells of K.

2. Now let M be a finite or infinite simplicial n-complex satisfying the conditions

(a) At most a finite number of cells of M are incident to any one cell of M;

(b) M is a simple n-circuit.[2]

3. It follows from (b) that M is identical with the sum of its closed n-cells. Suppose that $x^1, x^2, \cdots \to x$ where all the x's are points of M. Then if

[1] *Abstrakte kontinuierliche Gruppen* (Abh. math. Seminar, Hamburg, 5 (1927), pp. 15–32.)

[2] For definition see Lefschetz, Topology, pp. 46–7. (The complex L is to be taken to be zero.) The same definition holds equally well for infinite complexes. See chapter VII.

$\bar{E}^a, \bar{E}^b, \cdots, \bar{E}^c$ are those closed n-cells which contain x, it follows from §1 that almost all the x^i's lie in $\bar{E}^a + \cdots + \bar{E}^c$ and hence infinitely many $x^{i'}$s lie in some one of these cells.

4. As a consequence of (a), we may assume that the ordering $E^1, E^2, \cdots$ of the n-cells of M is such that, for each i, E^i has at least one face in common with at least one of the cells $E^1, \cdots, E^{i-1}$. In fact, choose E^1 arbitrarily. Since each $(n-1)$-cell is the face of two n-cells, we can choose an E^2 having a face common with E^1. Suppose our ordering proceeds as far as $E^1, \cdots, E^q$ where q is less than the total number of n-cells; its further progress will be balked only if each face of each E^i of the set $E^1, \cdots, E^q$ is also a face of another E^i of the set. But this would imply

$$E^1 + \cdots + E^q \to 0 \pmod 2$$

which contradicts the definition of simple circuit.

5. Let $e^1 = 0$ and e^i $(i = 2, 3, \cdots)$ be a definitely chosen face which E^i has in common with say $E^{i'}$ $(i' < i)$ and let

$$F^h = \sum_{i=1}^{h} (E^i + e^i);$$

for each h, F^h *is simply connected.* This is obvious for $h = 1$. Assume it for $h = m - 1$. A closed path in F^m, which begins and ends at a point O, say in E^1, can be deformed into a similar path in $F^{m-1} + e^m$, then to one in F^{m-1} and then, by hypothesis, to O. Hence F^m is simply connected.

6. Let C be a closed compact subset of M; then for a sufficiently large $\bar{n}$, $C \subseteq \bar{F}^{\bar{n}}$. For, C can meet at most a finite number of $\bar{E}^i$'s on account of §1 (Corollary). Hence for $\bar{n}$ large enough

$$C \subseteq \bar{E}^1 + \cdots + \bar{E}^{\bar{n}} = \bar{F}^{\bar{n}}.$$

7. Concerning the fundamental group G of M, we shall assume that it is abelian and has a finite basis, each element of which is of infinite order.

Let O be a definitely chosen point in M, say a point in E^1. We may take for the generators of G a set of closed paths

$$\omega_1, \cdots, \omega_m$$

each beginning and ending at O, and such that no linear combination of them can be deformed to O. The general group element of G is then $\lambda_1 \omega_1 + \cdots + \lambda_m \omega_m$, the λ's being arbitrary integers.

8. **The covering space.** We now recall briefly the definition of the universal covering space Λ of M.[3] The points of Λ are *pairs* of the form (x, w_x) where x is an arbitrary point of M, and w_x is an arbitrary path $\widehat{Ox}$. The points (x, w_x) and (y, w_y) are regarded as identical if, and only if, $x = y$ and $w_x - w_y = 0$; the latter relation means that the closed path $w_x - w_y$ is deformable to O. The point (x, w_x) is said to *cover* x, and the totality of points which cover x is identical with the set of points

$$(x, \lambda_1\omega_1 + \cdots + \lambda_m\omega_m + w^0)$$

where w^0 is an arbitrary but fixed path $\widehat{Ox}$ and the λ's are arbitrary integers. We shall denote this set by $\Lambda(x)$. If E is a point set in M, denote by $\Lambda(E)$ the set $\{\Lambda(x), x \in E\}$.

Let V be a simply connected set in M and w^0 a fixed path from O to a fixed point x^0 in V. Let y be an arbitrary point of V and u_y an arbitrary path $\widehat{x^0y}$. The point $(y, w^0 + u_y)$ of Λ is independent of the choice of u_y since V is simply connected. Consequently, the totality V_0 of these points is in (1, 1) correspondence with the points of V. In particular, if V is a closed p-cell of M, the correspondence may be thought of as defining a closed p-cell V_0 in Λ.

9. Suppose that w^0 is replaced by $\lambda_1\omega_1 + \cdots + \lambda_m\omega_m + w^0$. Denote the resulting set in Λ by $V_{\lambda_1 \cdots \lambda_m}$. Then if there is a point common to $V_{\lambda_1 \cdots \lambda_m}$ and $V_{\mu_1 \cdots \mu_m}$, it must be simultaneously of the forms

$$(x, \lambda_1\omega_1 + \cdots + \lambda_m\omega_m + w^0), \qquad (x, \mu_1\omega_1 + \cdots + \mu_m\omega_m + w^0)$$

which requires that $\lambda_i = \mu_i$ $(i = 1, \cdots, m)$. It follows that $\{V_{\lambda_1 \cdots \lambda_m}\}$ is a denumerable aggregate of mutually exclusive sets. In particular, if V is a closed p-cell of M, the sets $V_{\lambda_1 \cdots \lambda_m}$ are a denumerable set of mutually exclusive p-cells of Λ. These cells are said to "cover" V, and clearly $\Sigma\, V_{\lambda_1 \cdots \lambda_m} = \Lambda(V)$.

It follows readily from elementary consideration based on the preceding remarks that Λ is an infinite complex satisfying (a) (b) of §2. In particular Λ is the sum of its closed n-cells.

10. We shall call $\lambda = (O, \lambda_1\omega_1 + \cdots + \lambda_m\omega_m)$ a *lattice point* of Λ and $\lambda_1, \cdots, \lambda_m$ its *coördinates*. Often a lattice point will be denoted simply by its coördinates, and we shall write $\lambda = (\lambda_1, \cdots, \lambda_m)$. The *distance* between lattice points λ and μ will be defined by the euclidean formula

$$d(\lambda, \mu) = + \sqrt{(\lambda_1 - \mu_1)^2 + \cdots (\lambda_m - \mu_m)^2}.$$

11. Now let us choose a definite $\bar{n} > 0$ and consider the set $F^{\bar{n}}$. For the present we shall not specify $\bar{n}$ further than to agree that, when M is finite, $\bar{n}$ is to equal the total number of n-cells so that $M = \bar{F}^{\bar{n}}$. Let us denote the cell

[3] See, for example, Veblen, Analysis Situs (second edition), p. 152.

$F^{\bar{n}}_{\lambda_1 \cdots \lambda_m}$ (§9) by Ω_λ. Each Ω_λ contains its associated lattice point λ. Each Ω_λ is moreover the sum of a finite number of n-cells of Λ plus certain of their faces, and hence each $\bar{\Omega}_\lambda$ is the sum of a finite number of closed n-cells of Λ. Since $F^{\bar{n}}$ is simply connected, the sets Ω_λ are mutually exclusive (§9).

12. It is not necessarily true, however, that the $\bar{\Omega}_\lambda$'s are mutually exclusive. But since $\bar{\Omega}_0$ is a finite subcomplex, it will, on account of (a), §2, be met by at most a finite number of $\bar{\Omega}_\lambda$'s. Hence there exists a number A such that if $\bar{\Omega}_\lambda \bar{\Omega}_0 \neq 0$, $d(0, \lambda) < A$. Now $\bar{\Omega}_\lambda \bar{\Omega}_\mu$ is zero or not according as[4] $\bar{\Omega}_{\lambda-\mu} \bar{\Omega}_0$ is or is not, and since $d(\lambda - \mu, 0) = d(\lambda, \mu)$ we have the result,

13. There exists an A independent of λ and μ such that the relation $\bar{\Omega}_\lambda \bar{\Omega}_\mu \neq 0$ implies $d(\lambda, \mu) < A$.

13.1. Corollary. There exists an integer N such that at most N $\bar{\Omega}_\lambda$'s meet any one of them.

14. We shall study particularly the set $\Omega = \sum_\lambda \bar{\Omega}_\lambda = \overline{\Lambda(F^{\bar{n}})}$, an infinite subcomplex of Λ, identical with the sum of its closed n-cells. In case M is finite, $\Omega = M$ (§11).

Suppose $\lambda^1, \lambda^2, \cdots$ is an infinite sequence of distinct lattice points. *Then the set* $\Omega^0 = \bar{\Omega}_{\lambda 1} + \bar{\Omega}_{\lambda 2} + \cdots$ *is closed in* Ω. For, suppose $\tau^1, \tau^2, \cdots \to \tau$ where $\tau^i \in \Omega^0$ and $\tau \in \Omega$. Let $\phi_a, \phi_b, \cdots, \phi_c$ be the closed n-cells of Λ which contain τ. Then (§3) infinitely many τ^i's are contained in one of these cells, say ϕ_a. Since Ω^0 is the sum of its closed n-cells, and the τ^i's are in Ω^0, ϕ_a must be a cell of Ω^0. Hence $\tau \in \Omega^0$ and Ω^0 is closed.

15. If E is a closed set in M, then $\dim \Lambda(E) = \dim E$, where "dimension" is to be taken in the Menger-Urysohn sense.

Proof. We may write

$$E = \bar{E}^1 E + \bar{E}^2 E + \cdots$$

and each $\bar{E}^i E$ is closed. $\Lambda(\bar{E}^i E)$ is the sum of a denumerable number of non-overlapping closed sets, each homeomorphic with $\bar{E}^i E$. (See §§8–9) Hence[5] $\dim \Lambda(\bar{E}^i E) = \dim (\bar{E}^i E)$. Now $\dim (\bar{E}^i E) \leqq \dim E$; hence the dimension of $\Sigma\Lambda(\bar{E}^i E)$, that is, of $\Lambda(E)$ cannot be greater than that of E.[6] On the other hand, at least one set $\bar{E}^i E$—and hence at least one set $\Lambda(\bar{E}^i E)$—must be of same dimension as E.[7] Hence $\dim \Lambda(E) = \dim E$.

16. Let $\Omega^{(2)}, \Omega^{(3)}, \cdots$ be successive regular subdivisions of $\Omega = \Omega^{(1)}$, so chosen that mesh $\Omega^{(i)} \to 0$ with $1/i$. Denote the closed n-cells of $\Omega^{(h)}$ by $\Phi^h_1, \Phi^h_2, \cdots$.

Let D_r be a hypersphere of radius r in a cartesian S_m, center at the origin 0. Let $D = D_1$, and let $D^{(1)}, D^{(2)}, \cdots$ be successive simplicial subdivisions of D,

[4] $\lambda - \mu = (\lambda_1 - \mu_1, \cdots, \lambda_m - \mu_m)$.

[5] Menger, Dimensionstheorie (to be denoted hereafter by M.), p. 92.

[6] M., first theorem on p. 93.

[7] Otherwise $\dim (\bar{E}^i E) < \dim E$ (M., first theorem on p. 93).

each a regular subdivision of the preceding and such that mesh $D^{(i)} \to 0$. The subdivisions are to be cut out on D by flat spaces through 0. Two additional requirements are to be satisfied:

(a) In case $m > n$, no vertex of any subdivision is to lie in the space $x_{n+1} = \cdots = x_m = 0$.

(b) A ray from 0 through an arbitrary vertex of the system of subdivisions is to contain at least one and hence infinitely many lattice points of S_m.

The possibility of choosing the $D^{(i)}$'s to satisfy (b) becomes clear when we observe that the radial projections of the lattice points of S_m on D cover D densely.

17. We define certain subsets of S_m and Ω.

The closed cells of $D^{(h)}$ are Y_i^h $(i = 1, \cdots, i_h)$.

X_i^h consists of the points of S_m on, or exterior to D_h and contained in the set of rays subtended at 0 by Y_i^h.

$X^h = \sum_i X_i^h = D_h$ plus its exterior region.

$X_0^h =$ the interior region of D_h.

$\bar{X}_i^h$, $\bar{X}^h$, $\bar{X}_0^h$ are the sets of lattice points of S_m contained respectively in X_i^h, X^h, X_0^h; $\bar{X}^h$ and $\bar{X}_0^h$ are mutually exclusive and together contain all the lattice points of S_m.

$\tilde{\mathrm{X}}_i^h$ consists of those lattice points of Ω which have the same coördinates as lattice points of X_i^h; similarly define $\tilde{\mathrm{X}}^h$ and $\tilde{\mathrm{X}}_0^h$. Define further,

$$\mathrm{X}_i^h = \sum_\lambda \bar{\Omega}_\lambda \qquad (\lambda \in \tilde{\mathrm{X}}_i^h)$$

$$\mathrm{X}^h = \sum_\lambda \bar{\Omega}_\lambda \qquad (\lambda \in \tilde{\mathrm{X}}^h)$$

$$\mathrm{X}_0^h = \sum_\lambda \bar{\Omega}_\lambda \qquad (\lambda \in \tilde{\mathrm{X}}_0^h)$$

$\tilde{\mathrm{X}}_0^h + \tilde{\mathrm{X}}^h$ is the totality of lattice points of Ω; hence for each h,

$$\Omega = \mathrm{X}_0^h + \mathrm{X}^h = \mathrm{X}_0^h + \sum_i \mathrm{X}_i^h .$$

18. A given point π of Ω is contained in at most a finite number of the sets $\mathrm{X}^1, \mathrm{X}^2, \cdots$.

For, suppose on the contrary that $\pi \in \mathrm{X}^{h_i}$ $(i = 1, 2, \cdots)$. Then we can write

$$\pi \in \bar{\Omega}_{\lambda^i} \subset \mathrm{X}^{h_i} \qquad (i = 1, 2, \cdots)$$

since each X is a sum of $\bar{\Omega}_\lambda$'s. From the definition of the X^h's it is clear that $d(0, \lambda^i) \to \infty$ and hence infinitely many of the points λ^i are distinct; hence infinitely many distinct $\bar{\Omega}_{\lambda^i}$'s contain π, which contradicts 13.1.

19. Each X_0^h is the sum of a finite number of closed n-cells, Φ_i^h of $\Omega^{(h)}$. Let the lower indices of the Φ_i^h's be rearranged so that for each h,

$$X_0^h = \Phi_{i_{h+1}}^h + \Phi_{i_{h+2}}^h + \cdots + \Phi_{j_h}^h$$

and let

$$\Psi_i^h = X_i^h (i = 1, \cdots i_h) \, ; = \Phi_i^h (i = i_h + 1, \cdots j_h) \, . \tag{1}$$

It will be understood from now on that any Φ_i^h mentioned from now on will be one of those appearing in (1).

For each h, the sets $\Psi_1^h, \Psi_2^h, \cdots$ are closed (for the X's this follows from 14) and their sum is Ω.

20. Every infinite product $\Psi_{i_1}^1 \Psi_{i_2}^2 \cdots$ is either zero or a single point.

Proof. In case the product contains infinitely many Φ's, our assertion follows from 16. If there are infinitely many X's, the product is zero by 18.

21. If $\bar{\Omega}_\lambda, \bar{\Omega}_\mu$ meet Φ_a^h, Φ_b^k respectively and, if $\Phi_a^h \Phi_b^k \neq 0$, then $d(\lambda, \mu) < 3A$ where A is the constant of 13.

Proof. We can choose lattice points λ', μ' in Ω such that

$$\Phi_a^h \subset \bar{\Omega}_{\lambda'}, \qquad \Phi_b^k \subset \bar{\Omega}_{\mu'} .$$

Hence $\bar{\Omega}_{\lambda'} \bar{\Omega}_{\mu'} \neq 0$ and by 13 $d(\lambda', \mu') < A$. Since, by hypothesis, $\bar{\Omega}_\lambda \bar{\Omega}_{\lambda'} \neq 0$, $\bar{\Omega}_\mu \bar{\Omega}_{\mu'} \neq 0$, we have $d(\lambda, \lambda') < A$, $d(\mu, \mu') < A$ and from these inequalities follows the desired one.

22. Let $'X_i^h$ denote X_i^h plus all the Φ_i^h's which meet X_i^h. Then if $'X_i^h \, 'X_j^k \neq 0$, there exist lattice points λ, μ in X_i^h, X_j^k respectively such that $d(\lambda, \mu) < 3A$.

Proof. If $'X_i^h$ and $'X_j^k$, meet, we can choose Φ_a^h and Φ_b^k meeting X_i^h, X_j^k respectively, and meeting each other. Since Φ_a^h meets X_i^h, Φ_a^h will meet say $\bar{\Omega}_\lambda$ where $\bar{\Omega}_\lambda \subset X_i^h$. Similarly, Φ_b^k meets say $\bar{\Omega}_\mu$, $\bar{\Omega}_\mu \subset X_j^k$. Our assertion now follows from 21.

23. If $y^1, \cdots, y^q$ are arbitrary points in $Y_{i_1}^{n\,1}, \cdots, Y_{i_q}^{n\,q}$ respectively, and if $Y_{i_1}^{n\,1} \bullet \cdots \bullet Y_{i_q}^{n\,q} = 0$, then $\inf \left(\sum_{i,j=1}^q d(y^i, y^j) \right) > 0$.

Proof. $\Sigma \, d(y^i, y^j)$ is a non-negative continuous function of the points $y^1, \cdots, y^q$ defined over a closed bounded subset of S_m; hence it assumes its lower bound, and this lower bound can only be zero when $y^1 = \cdots = y^q$; this, however, would imply $Y_{i_1}^{n\,1} \bullet \cdots \bullet Y_{i_q}^{n\,q} \neq 0$.

23.1. Let $x^{1r}, \cdots, x^{qr}$ $(r = 1, 2, \cdots)$ be infinite sequences of points in $X_{i_1}^{n\,1}, \cdots, X_{i_q}^{n\,q}$ respectively, and such that

$$\lim_{r \to \infty} d(0, x^{hr}) = \infty \qquad (h = 1, \cdots, q) .$$

Then if $Y_{i_1}^{n\,1} \bullet \cdots \bullet Y_{i_q}^{n\,q} = 0$, $\lim_{r \to \infty} \sum_{h,j=1}^q d(x^{hr}, x^{jr}) = \infty$.

Proof. $Y^{n}_{i\,1} \bullet \cdots \bullet Y^{n}_{i\,q} = 0$ implies $X^{n}_{i\,1} \bullet \cdots \bullet X^{n}_{i\,q} = 0$ (§17). Let y^{rs} be the radial projection of x^{rs} on D. Then $y^{rs} \in Y^{n}_{i\,s}$. Now suppose that for every $B > 0$, there is an arbitrarily large r for which $\sum_{h,k} d(x^{rh}, x^{rk}) < B$. This inequality implies $d(x^{rh}, x^{rk}) < B$, and hence for large values of r, the radial projections of x^{rh}, x^{rk} on D are near together because, by hypothesis, x^{rh}, x^{rk} are at large distances from 0. Hence $\inf_{r} (\sum_{h,k} d(y^{rh}, y^{rk})) = 0$ which by 23 implies $Y^{n}_{i\,1} \bullet \cdots \bullet Y^{n}_{i\,q} \neq 0$; this is a contradiction.

24. If $\lambda^{r1}, \cdots, \lambda^{rq}$ $(r = 1, 2 \cdots)$ are infinite sequences of lattice points of Ω in $X^{n}_{i\,1}, \cdots, X^{n}_{i\,q}$ respectively and, if for each h $\lim_{r\to\infty} (0, \lambda^{rh}) = \infty$, and if $Y^{n}_{i\,1} \bullet \cdots \bullet Y^{n}_{i\,q} = 0$, then $\lim_{r\to\infty} \sum_{h,k=1}^{q} d(\lambda^{rh}, \lambda^{rk}) = \infty$.

Proof. Let x^{rs} be that lattice point of $X^{n}_{i\,s}$ which has the same coördinates as λ^{rs}. Then since distances are preserved under this correspondence, our assertion follows from 23.

25. If $Y^{n}_{i\,1} \bullet \cdots \bullet Y^{n}_{i\,q} = 0$, there exists an R such that $X^{r}X^{n}_{i\,1} \bullet \cdots \bullet X^{n}_{i\,q} = 0$ for $r > R$.

Proof. Suppose that for a certain fixed r the product in question is $\neq 0$; let τ be a point in the product. Since the X's are sums of $\bar{\Omega}_{\lambda}$'s we can write

$$\tau \in \bar{\Omega}_{\lambda 0} \subset X^{r}, \qquad \tau \in \bar{\Omega}_{\lambda 1} \subset X^{n}_{i\,1}, \cdots, \tau \in \bar{\Omega}_{\lambda q} \subset X^{n}_{i\,q}$$

and by 13,

$$d(\lambda^{0}, \lambda^{h}) < A, \qquad d(\lambda^{h}, \lambda^{k}) < A \qquad (h, k = 1, \cdots, q)$$

so that

$$\sum_{h,k=1}^{q} d(\lambda^{h}, \lambda^{k}) < q^{2}A . \tag{1}$$

Now suppose, contrary to conclusions, that the r chosen above can be arbitrarily large. We could then choose sequences of lattice points

$$\lambda^{r0}, \lambda^{r1}, \cdots, \lambda^{rq} \qquad (r = r_1, r_2, \cdots, \text{an increasing sequence})$$

contained respectively in $X^{r}, X^{n}_{i\,1}, \cdots, X^{n}_{i\,q}$. In each case, we have

$$\sum_{h,k=1}^{q} d(\lambda^{r_i h}, \lambda^{r_i k}) < q^{2}A . \tag{2}$$

On the other hand, since $\lambda^{r0} \in X^{r}$, we have $\lim_{i\to\infty} d(0, \lambda^{r_i 0}) = \infty$; and, since $d(\lambda^{r_i 0}, \lambda^{r_i h}) < A$, we have $\lim_{i\to\infty} d(0, \lambda^{r_i h}) = \infty$ which contradicts (2).

25.1. Corollary. If $X^r X^{n_1}_{i_1} \bullet \cdots \bullet X^{n_q}_{i_q} \neq 0$ for arbitrarily large values of r, then $Y^{n_1}_{i_1} \bullet \cdots \bullet Y^{n_q}_{i_q} \neq 0$.

25.2. If $Y = Y^{n_1}_{i_1} \bullet \cdots \bullet Y^{n_q}_{i_q} \neq 0$, then there exist arbitrarily large values of r for which $X^r X^{n_1}_{i_1} \bullet \cdots \bullet X^{n_q}_{i_q} \neq 0$.

Proof. Y is a closed cell of some subdivision of D. Let y be a vertex of Y. Then by 16 (b) the ray $0y$ contains an infinite sequence of lattice points, and each of these from a certain rank on is contained in $X^{n_1}_{i_1} \cdots X^{n_q}_{i_q} = X$; hence, for every given r, almost all of them are contained in $X^r X$. The corresponding lattice points of Ω are contained in $X^r X^{n_1}_{i_1} \cdots X^{n_q}_{i_q}$ and our conclusion follows.

26. **Fundamental sequences.**[8] Let $\alpha_h = \{a, b, \cdots, c\}$ be a finite set of distinct integers chosen from the set

$$1, \cdots, i_h, \cdots, j_h \qquad (\S 19)$$

and let the product $\Psi^h_a \Psi^h_b \bullet \cdots \bullet \Psi^h_c$ be denoted by $\Psi^h_{\alpha_h}$. We shall call the infinite sequence of sets

$$\alpha_1, \alpha_2, \cdots \tag{1}$$

a fundamental sequence (f.s.) if

(a) $\Psi^1_{\alpha_1} \Psi^2_{\alpha_2} \bullet \cdots \bullet \Psi^h_{\alpha_h} \neq 0$ for $h = 1, 2, \cdots$.

(b) Each α_i is "maximal" relative to property (a); that is, (a) fails to hold if any α_i is replaced by an α'_i such that $\alpha_i \subset \alpha'_i$.

Each point α of Ω is associated with a unique f.s. in the following way. Let $\Psi^h_{a_h}, \cdots, \Psi^h_{c_h}$ be all the Ψ^h_i's which contain α. Let $\alpha_h = \{a_h, \cdots, c_h\}$. Then $\alpha \in \Psi^1_{\alpha_1} \Psi^2_{\alpha_2} \bullet \cdots$, and the α_i's therefore satisfy (a). For h sufficiently large, the factors of $\Psi^h_{\alpha_h}$ consist only of Φ^h_i's, by 18, and, since mesh $\Omega^{(h)} \to 0$, we have $\alpha = \Psi^1_{\alpha_1} \Psi^2_{\alpha_2} \bullet \cdots$. Now if $\alpha'_i \supset \alpha_i$, some factor say Ψ^i_a of $\Psi^i_{\alpha'_i}$ will fail to contain α, and consequently $\Psi^1_{\alpha_1} \bullet \cdots \bullet \Psi^i_{\alpha_i} \bullet \cdots = (\Psi^i_a \bullet \cdots)(\Psi^1_{\alpha_1} \Psi^2_{\alpha_2} \bullet \cdots) = 0$ and hence (b) is satisfied. Thus $\alpha_1, \alpha_2, \cdots$ is a f.s.; we shall call it the "f.s. of α."

If (1) is a f.s. and if at least one of its sets, say α_h, contains an integer, say a, of the set $i_h + 1, \cdots, j_h$, then (1) is the f.s. of a unique point of Ω. For, if $k > h$

$$\Psi^k = \Psi^1_{\alpha_1} \cdots \Psi^k_{\alpha_k} \subseteq \Phi^h_a$$

and hence Ψ^k is closed and compact. Since $\Psi^1 \supseteq \Psi^2 \supseteq \cdots$, it follows that $\Psi^1_{\alpha_1} \Psi^2_{\alpha_2} \bullet \cdots = \Psi^1 \Psi^2 \bullet \cdots = \Psi^k \Psi^{k+1} \bullet \cdots \neq 0$. Let α be a point of $\prod_i \Psi^i_{\alpha_i}$. For p sufficiently large, no X^p_i contains α (by 18) and hence the factors of $\Psi^h_{\alpha_h}$ consist

[8] See for example M., p. 182 et seq. and p. 280 et seq. We use the expression "fundamental sequence" instead of "chain" (Kette) since the latter word has been appropriated for use in combinatorial topology.

exclusively of Φ_i^h's when $h > p$. Hence $\prod_i \Psi_{\alpha_i}^i = \alpha$. Finally, the factors of each $\Psi_{\alpha_h}^h$ are all the Ψ_i^h's which contain α, otherwise some α_h would not be maximal.

From the preceding paragraph it follows that a f.s. which corresponds to no point of Ω will involve only integers of the sets $1, \cdots, i_h$. Such f.s.'s will be called *ideal*. If $\alpha_1, \alpha_2, \cdots$ is ideal, we will have

$$\Psi_{\alpha_1}^h = X_{\alpha_h}^h = X_a^h X_b^h \bullet \cdots \bullet X_c^h, \{a, b, \cdots, c\} = \alpha_h .$$

27. Fundamental sequences can of course be defined equally well for D with aid of the sets Y_i^h. It is easy to see that since D is compact, the correspondence between points of D and f.s.'s of D is (1, 1). We shall show further that *every ideal f.s. of Ω is identical with a f.s. of D and vice versa.*

Proof. Let $\alpha_1, \alpha_2, \cdots$ be an ideal f.s. of Ω. Then for every r and $h (r \geqq h)$

$$X_{\alpha_1}^1 \bullet \cdots \bullet X_{\alpha_h}^h \bullet \cdots \bullet X_{\alpha_r}^r \neq 0 .$$

Since $X_{\alpha_r}^r \subset X^r$, we have

$$X^r X_{\alpha_1}^1 \bullet \cdots \bullet X_{\alpha_h}^h \bullet \cdots \bullet X_{\alpha_r}^r \neq 0 \qquad (r \geqq h).$$

Hence $X^r X_{\alpha_1}^1 \bullet \cdots \bullet X_{\alpha_h}^h \neq 0$ for $r = h, h + 1, \cdots$. Hence by 25.1, $Y_{\alpha_1}^1 \bullet \cdots \bullet Y_{\alpha_h}^h \neq 0$, $(h = 1, 2, \cdots)$. To see that each α_i is maximal for D, suppose $\alpha_p' \supset \alpha_p$. Then since $X_{\alpha_1}^1 \bullet \cdots \bullet X_{\alpha_p}^p \bullet \cdots X_{\alpha_h}^h = 0$ for some h, (α_p being maximal for Ω), we have $Y_{\alpha_1}^1 \bullet \cdots \bullet Y_{\alpha_p'}^p \bullet \cdots \bullet Y_{\alpha_h}^h = 0$ by 25.2. Hence α_p is maximal for D.

Suppose, conversely, that $\alpha_1, \alpha_2, \cdots$ is a f.s. of D. Then since $Y_{\alpha_1}^1 \bullet \cdots \bullet Y_{\alpha_h}^h \neq 0$ $(h = 1, 2, \cdots)$ we have $X_{\alpha_1}^1 \cdots X_{\alpha h}^h \neq 0$ by 25.2. Let $\alpha_p' \supset \alpha_p$. Then, by hypothesis, $Y_{\alpha_1}^1 \bullet \cdots \bullet Y_{\alpha_p'}^p \bullet \cdots \bullet Y_{\alpha_h}^h \neq 0$ for some h; hence by 25 we can choose an integer $r > h$ such that

$$X^r X_{\alpha_1}^1 \bullet \cdots \bullet X_{\alpha_p'}^p \bullet \cdots \bullet X_{\alpha_h}^h = 0 .$$

Then

$$X_{\alpha_1}^1 \bullet \cdots \bullet X_{\alpha_p'}^p \bullet \cdots \bullet X_{\alpha_r}^r \subseteq X_{\alpha_1}^1 \bullet \cdots \bullet X_{\alpha_p'}^p \bullet \cdots \bullet X_{\alpha_h}^h X_{\alpha_r}^r$$

$$\subseteq X_{\alpha_1}^1 \bullet \cdots \bullet X_{\alpha_p'}^p \bullet \cdots \bullet X_{\alpha_h}^h X^r = 0 .$$

Hence α_p is maximal for Ω.

28. Let us consider the ideal fundamental sequences of Ω as defining a set of ideal points which are to be added to Ω forming an enlarged space which will be denoted by $\bar{\Omega}$. Each point of $\bar{\Omega}$ now has a unique f.s. In the following lemmas it will be convenient to use Greek letters when referring to points or sequences of $\bar{\Omega}$

and Latin letters for those of D. Thus the relation $\alpha = \{\alpha_1, \alpha_2, \cdots\}$ implies that α is a point of $\bar{\Omega}$ and its f.s. is $\alpha_1, \alpha_2, \cdots$. For D we would have written $a = \{a_1, a_2, \cdots\}$.

29. We define an h-neighborhood[9] $U_h(\alpha)$ to be the totality of points $\beta = \{\beta_1, \beta_2, \cdots\}$ such that $\beta_i \subseteq \alpha_i$ $(i = 1, \cdots, h)$. $U_h(a)$ is defined in the same way. It is easy to verify the fact that $U_h(a)$ is the sum Y of the sets Y_i^h which contain a, minus the boundary points of Y. We shall show that, with the topology defined by its h-neighborhoods, $\bar{\Omega}$ is compact.

30. The point $\alpha = \{\alpha_1, \alpha_2, \cdots\}$ is contained in Ψ_i^h if h is an integer of the set α_h. This follows from §27 in the case of a Φ_i^h, in the case of a X_i^h we shall take this statement to be a definition.

Let $C_h(\alpha)$ be the sum of all the Ψ_i^h's which contain α, and let $C_h'(\alpha)$ be the sum of all the Φ_i^h's which meet $C_h(\alpha)$. Obviously $C_h(\alpha) \subseteq C_h'(\alpha)$.

30.1. $U_h(\alpha) \subseteq C_h(\alpha)$.

Proof. If ξ is a point of $U_h(\alpha)$ then $\xi_i \subseteq \alpha_i$ $(i = 1, \cdots, h)$ and hence each factor of $\Psi_{\xi_h}^h$ is also a factor of $\Psi_{\alpha_h}^h$ and therefore contains α. Now $C_h(\alpha)$ contains all the Ψ_i^h's which contain α; hence $\Psi_{\xi_h}^h \subseteq C_h(\alpha)$ and hence $\xi \in C_h(\alpha)$.

31. Suppose that the points α and a of $\bar{\Omega} - \Omega$ and D have identical fundamental sequences. If $C_h(\alpha) = \mathrm{X}_a^h + \mathrm{X}_b^h + \cdots + \mathrm{X}_c^h$, then $C_h(a) = Y_a^h + Y_b^h + \cdots + Y_c^h$ and conversely.

Proof. X_i^h contains α or not according as Y_i^h contains a or not (see §27).

32. If C_h (a) $= Y_a^h + \cdots + Y_c^h$, then let $C_h^{\mathrm{X}}(a) = X_a^h + \cdots + X_c^h$. $C_h(a)$ is the radial projection of $C_h^{\mathrm{X}}(a)$ on D. Under the hypothesis of 31, *the lattice points in* $C_h(\alpha)$ *are in* (1, 1) *correspondence with those in* $C_h^{\mathrm{X}}(a)$, corresponding points having the same coördinates. This follows from 31 and the definition of the sets involved.

33. If $a \neq b$ and l^s, m^s $(s = 1, 2, \cdots)$ are lattice points in $C_s^{\mathrm{X}}(a)$, $C_s^{\mathrm{X}}(b)$, respectively, then $\lim_{s \to \infty} d(l^s, m^s) = \infty$.

For, it will be seen that l^s and m^s become increasingly distant from 0 as $s \to \infty$, while their radial projections on D converge to a and b respectively.

34. If $C_h'(\alpha)\, C_h'(\beta) \neq 0$, α and β being ideal, then lattice points λ, μ can be chosen in $C_h(\alpha)$, $C_h(\beta)$ such that $d(\lambda, \mu) < 3A$.

Proof. $C_h(\alpha)$ and $C_h(\beta)$ are sums of X_i^h's; clearly $C_h'(\alpha)$ and $C_h'(\beta)$ are the sums

[9] M., p. 184.

of the corresponding $'X_i^h$'s (see 22). Consequently we can choose $'X_i^h$, $'X_j^h$ such that

$$'X_i^h \subseteq C_h'(\alpha), \qquad 'X_j^h \subseteq C_h'(\beta), \qquad 'X_i^h\, 'X_j^h \neq 0.$$

The conclusion now follows from 22.

35. If $\alpha \neq \beta$, there exists an $\bar{s}$ such that $C_s'(\alpha)\, C_s'(\beta) = 0$ for $s > \bar{s}$.

Proof. Suppose first that α and β are both ideal. If the statement is false, we would have for some sequence $s_1 < s_2 < \cdots$,

$$C_s'(\alpha)\, C_s'(\beta) \neq 0 \qquad (s = s_1, s_2, \cdots).$$

By 34 there are lattice points λ^s, μ^s in $C_s(\alpha)$, $C_s(\beta)$ with $d(\lambda^s, \mu^s) < 3A$. The corresponding lattice points l^s, m^m in S_m are in $C_h^X(a)$, $C_h^X(b)$ by 31, and $d(l^s, m^s) < 3A$ $(s = s_1, s_2, \cdots)$ which contradicts 33.

The proof in the case that one or both of the points are non-ideal offers no difficulty.

36. If α and β are ideal and if $C_h(\alpha)\, C_h(\beta) = 0$, and if λ^s, μ^s $(s = 1, 2 \cdots)$ are lattice points in $C_h(\alpha)$, $C_h(\beta)$ such that $\lim_{s\to\infty} d(0, \lambda^s) = \lim_{s\to\infty} d(0, \mu^s) = \infty$, then $\lim_{s\to\infty} d(\lambda^s, \mu^s) = \infty$.

Proof. Suppose $C_h(\alpha)$ and $C_h(\beta)$ each consist of a single X, say $C_h(\alpha) = X_i^h$, $C_h(\beta) = X_j^h$. Then $X_i^h X_j^h = 0$ and therefore $Y_i^h Y_j^h = 0$ by 25.2 and the result follows from 24. The general case offers no additional difficulty.

37. If $C_h(\alpha)\, C_h(\beta) = 0$, then there exists an $\bar{s}$ such that no X_i^s can meet both $C_h(\alpha)$ and $C_h(\beta)$ if $s > \bar{s}$.

Proof. There is no difficulty when at least one of the points α, β is non-ideal. Assume both to be ideal. The hypothesis $C_h(\alpha)\, C_h(\beta) = 0$ obviously implies that $\alpha \neq \beta$. Moreover, $C_h(a)\, C_h(b) = 0$, for if Y_i^h (say) in $C_h(a)$ meets Y_j^h in $C_h(b)$, then (25) X_i^h meets X_j^h and by 31 these two X's are in $C_h(\alpha)$, $C_h(\beta)$ respectively, so that we would have $C_h(\alpha)\, C_h(\beta) \neq 0$. It follows that there is a $t > 0$ such that $d(p, q) > t$ when p and q are arbitrary points of $C_h(a)$ and $C_h(b)$. Now suppose, contrary to conclusion, that for $s = s_1, s_2, \cdots$, (an increasing sequence) $X_{i_s}^s$ meets both $C_h(\alpha)$ and $C_h(\beta)$. Since the C's are sums of X's and the latter are sums of $\bar{\Omega}$'s, we can choose lattice points λ^s, μ^s, σ^s, τ^s with

$$\lambda^s \in X_{i_s}^s, \qquad \mu_s \in X_{i_s}^s, \qquad \sigma^s \in C_h(\alpha), \qquad \tau^s \in C_h(\beta)$$

and hence $d(\lambda^s, \sigma^s) < 3A$, $d(\mu^s, \tau^s) < 3A$ by 13. Let l^s, m^s, s^s, t^s be the corresponding lattice points of S_m and let l_D^s, m_D^s be the radial projections of l^s, m^s on D. Then since $l^s \in X_{i_s}^s$ and $m^s \in X_{i_s}^s$, we have $d(l_D^s, m_D^s) \to 0$ when $s = s_i$, $i \to \infty$. Since $d(s^s, l^s) < 3A$ and $d(t^s, l^s) < 3A$, it is clear that $d(t_D^s, s_D^s) \to 0$, which is impossible since t_D^s, s_D^s are in $C_h(a)$, $C_h(b)$ respectively.

38. If $\alpha \neq \beta$, there exist integers r and $\bar{s}$ such that if $\xi = \{\xi_1, \xi_2, \cdots\}$ and $\eta = \{\eta_1, \eta_2, \cdots\}$ are points in $U_r(\alpha)$, $U_r(\beta)$ respectively, then for every $s > \bar{s}$, ξ_s and η_s have no common element.

Proof. Choose an r by 35 such that $C'_r(\alpha)\, C'_r(\beta) = 0$. By 37 choose $\bar{s} \geqq r$ such that no X^s_i will meet $C_r(\alpha)$ and $C_r(\beta)$ when $s > \bar{s}$. We assert now that no Φ^s_i will meet both $C_r(\alpha)$ and $C_r(\beta)$ when $s > \bar{s}$; for, if so, then some Φ^r_i would also meet both sets, since every Φ^s_i is contained in a Φ^r_i if $r \leqq s$. It would follow that $C'_r(\alpha)\, C'_r(\beta) \neq 0$ contrary to choice of r. Thus, no Ψ^s_i $(s > \bar{s})$ meets $C_r(\alpha)$ and $C_r(\beta)$ and hence by 30.1 no Ψ^s_i meets both $U_r(\alpha)$ and $U_r(\beta)$ when $s > \bar{s}$. Hence no Ψ^s_i can contain both ξ and η when $s > \bar{s}$ and hence ξ_s and η_s $(s > \bar{s})$ can have no common integer.

39. As a consequence of the preceding result, we can now assert that with the topology defined by the h-neighborhoods, $\bar{\Omega}$ *is a compact separable space.* This follows from the theory of fundamental sequences, particularly, the work of Alexandroff.[10] It is clear that the topology which the U_h's define on Ω is identical with that which exists a priori.

Let the totality of ideal points be denoted by Δ; by 27 there is a (1, 1) correspondence $D \leftrightarrow \Delta$ such that corresponding points have identical fundamental sequences. Now consider Δ as a subspace of $\bar{\Omega}$. A set of neighborhoods for Δ is then $\{\Delta U_h(\alpha)\}$ $(h = 1, 2, \cdots, \alpha \in \Delta)$. Let a be the point of D corresponding to α, so that $a_i = \alpha_i$ $(i = 1, 2, \cdots)$. An arbitrary point ξ of $\Delta U_h(\alpha)$ is ideal and hence corresponds to a point x in D, and since $x_i = \xi_i \subseteq \alpha_i = a_i$, we have $x \in U_h(a)$. Conversely, every point in $U_h(a)$ corresponds to one in $U_h(\alpha)$. Thus neighborhoods correspond to neighborhoods, and $\Delta \leftrightarrow D$ is a homeomorphism.

In case the original complex M is finite so that $M = \Lambda$ (§14) we have now the theorem,

If the fundamental group of M is abelian and generated by m independent elements of infinite order, then the universal covering space can be "closed" in such a way that the ideal set is homeomorphic with an $(m - 1)$-sphere.

40. We now examine the convergence relations between Ω and Δ. Denote by $U^r_h(a)$ those points of the ray set subtended by $U_h(a)$ which are on or exterior to D_r (§16).

40.1. If $r \geqq h$ and if a, α are corresponding points under $D \leftrightarrow \Delta$, then for every lattice point in $U^r_h(a)$, there is one in $U_h(\alpha)$ having the same coördinates.

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[10] See M., p. 182 et seq. The stated property of $\bar{\Omega}$ is a result of conditions a) b) c) d) of M., p. 187, which hold for the fundamental sequences of Ω and their finite segments. a) and b) are satisfied trivially; c) follows readily for segments of non-ideal f.s.'s; for ideal f.s.'s use the fact that they are in (1, 1) correspondence with the f.s.'s of D; d) is precisely our result 38.

Proof. Let u and n be integers > 0 and such that $n \leqq u$. If $\alpha_n = a_n = (p, q, \cdots r)$, then

$$U_n(a) = Y_p^n + Y_q^n + \cdots + Y_r^n - \text{(boundary points)} \tag{1}$$

and

$$U_n^u(a) \subseteq X_p^n + X_q^n + \cdots + X_r^n \tag{2}$$

and hence if l is a lattice point of $U_n^u(a)$, l is contained in certain of the X's, say $X_s^n, \cdots, X_t^n$ appearing in (2). The set $(s, \cdots, t)$ is a subset of $(p, \cdots, r)$ and the corresponding X's are the only X's which contain l, since the radial projection of l on D falls in $U_n(a)$ and is therefore contained in no Y_i^n other than those in (1). It follows that if λ is the lattice of Ω corresponding to l, then λ is contained in each of the sets $X_s^n, \cdots, X_t^n$ and in no other X_i^n. Since λ is in X^n, it is in no Φ_i^k if $k \leqq n$ (see 17). Hence in the f.s. $\lambda = \{\lambda_1, \lambda_2, \cdots\}$, we have $\lambda_n = (s, \cdots, t)$ and hence $\lambda_n \subseteq \alpha_n$. To summarize, if $l \in U_n^u(a)$, u being any integer $\geqq n$, then $\lambda_n \subseteq \alpha_n$.

Now let r be an integer $\geqq h$. It follows from definitions that

$$U_h^r(a) \subseteq U_h^h(a) \subset U_{h-1}^{h-1}(a) \cdots \subset U_1^1(a).$$

Hence if $l \in U_h^r(a)$, then also $l \in U_n^n(a)$, $n = 1, \cdots, h$. Hence, by the result above, $\lambda_n \subseteq \alpha_n$ $(n = 1, \cdots, h)$ and $\lambda \in U_h(\alpha)$.

41. In the following lemmas we shall denote by λ, l corresponding lattice points in Ω, S_m (equal coördinates); by α, a corresponding points of Δ, D (equal f.s.'s); and by l_D the radial projection of l on D.

41.1. A necessary and sufficient condition that $\lambda^1, \lambda^2, \cdots \to \alpha$ is that (a) $d(0, \lambda^i) \to \infty$ and (b) that $l_D^i \to a$.

Proof. Suppose $\lambda^i \to \alpha$. Then for each h, almost all the λ^i's are in $C_h(\alpha)$ (30.1). Hence almost all the l^i's are in $C_h^X(a)$ (32). Hence almost all the l_D^i's are in $C_h(l)$ and hence $l_D^i \to l$. Obviously $d(0, \lambda^i) = d(0, l^i) \to \infty$.

Suppose now that (a) and (b) are satisfied. Then for each h, almost all the l^i's are in $U_h^r(a)$ for every $r \geqq h$. Hence (40.1) almost all the λ^i's are in $U_h(\alpha)$.

42. If $\lambda^1, \lambda^2, \cdots \to \alpha$ and if $\mu^1, \mu^2, \cdots$ are lattice points and K a constant such that $d(\mu^i, \lambda^i) < K$, then $\mu^i \to \alpha$.

Proof. By 14.1, $l_D^i \to a$, $d(0\ l^i) \to \infty$. Moreover, $d(l^i, m^i) < K$, and hence it is clear that $m_D^i \to a$ and $d(0, m^i) \to \infty$. Hence by 41.1, $\mu^i \to \alpha$.

43. If $\lambda \in U_h(\alpha)$ (α ideal) then $\Omega_\lambda \subset U_h(\alpha)$.

Proof. By hypothesis, $\lambda_q \subseteq \alpha_q$, $q = 1, \cdots, h$. Since α_q consists only of integers of the set $1, \cdots, i_q$, the same is true of λ_q $(q = 1, \cdots, h)$. Hence λ is contained in no Φ_i^q's if $q \leqq h$. Therefore, $\Psi_{\lambda_q}^q$ $(q \leqq h)$ is the product of all the

X_j^q's containing λ, i.e., $\Psi_{\lambda_q}^q = X_{\lambda_q}^q$. Since each X_j^q is the sum of $\bar{\Omega}_\mu$'s and since $\lambda \in X_{\lambda_1}^1 \bullet \cdots \bullet X_{\lambda_h}^h$, we have

$$\bar{\Omega}_\lambda \subseteq X_{\lambda_1}^1 \bullet \cdots \bullet X_{\lambda_h}^h \subseteq X^h. \tag{1}$$

Moreover, no point of $\bar{\Omega}_\lambda$ can be in any X_i^q $(1 \leqq q \leqq h)$ other than the factors of the $X_{\lambda_i}^i$'s in (1). For, if one point of $\bar{\Omega}_\lambda$ is in X_i^q, so is every point, λ in particular. Finally, no Ω_τ in X^h can meet any $\Phi_i^q (1 \leqq q \leqq h)$ (see 17 and 19). Hence an arbitrary point π of Ω_λ is contained in precisely the same Ψ_i^q's $(q = 1, \cdots, h)$ as λ is and hence $\pi \in U_h(\alpha)$.

44. If $\lambda^1, \lambda^2, \cdots \to \alpha$ and if ξ^i is a point of Ω_{λ^i}, then $\xi^i \to \alpha$.

Proof. $U_h(\lambda)$ contains almost all the λ^i's and hence by 43, almost all the ξ^i's.

45. If $\lambda^1, \lambda^2, \cdots \to \alpha$ and if $\xi^i \in \bar{\Omega}_{\lambda^i}$, then $\xi^i \to \alpha$.

Proof. We shall use the fact that $\bar{\Omega}$ is metric (theorem of Urysohn). For each i, choose in Ω_{λ^i} a point σ^i within a distance $1/i$ from ξ^i. Then $\sigma^i \to \alpha$ by 44, and hence $\xi^i \to \alpha$.

46. **The space Ω^*.** We shall now assume that $n < m$, Let Ω^* be the flat n-space in S_m defined by

$$x_{n+1} = \cdots = x_m = 0.$$

Ω^* intersects $D^{(h)}$ in a (non-simplicial) subdivision $D^{*(h)}$ of the $(n-1)$-sphere D^*. Each closed $(n-1)$-cell of $D^{*(h)}$ is the intersection of a Y_i^h with Ω^*. We may assume that the lower indices of the Y_i^h's are so arranged that the $(n-1)$-cells of $D^{*(h)}$ are

$$Y_i^{*h} = \Omega^* Y_i^h \qquad (i = 1, \cdots, i_h^* \leqq i_h).$$

We now refer to the property (a) of 16. As a result of this, together with the fact that the cells of $D^{(h)}$ are intersections of D by flat spaces through 0, it becomes apparent that Ω^* either fails to meet a given Y_i^h or else intersects it in a closed $(n-1)$-cell. As a result, the Y_i^{*h}'s which contain a given point p of D^* have the same lower indices as do the Y_i^h's which contain p and therefore, if $\{p\}_{D^*}$ and $\{p\}_D$ are the fundamental sequences of p regarded as a point of D^* and D respectively, then $\{p\}_{D^*} = \{p\}_D$ for every p in D^*.

47. Let $X_i^{*h} = \Omega^* X_i^h$ $(i = 1, \cdots i_h^*)$ and $X_0^{*h} = \Omega^* X_0^h$. X_0^{*h} consists of the points of Ω^* interior to D_h. Now subdivide $\bar{X}_0^{*h}$ into a finite number of closed n-cells F_i^{*h} $(i = i_h^* + 1, \cdots, j_h^*)$ of diameter $< 1/h$ and let

$$G_i^{*h} = X_i^{*h} \ (i = 1, \cdots, i_h^*); = F_i^{*h} \ (i = i_h^* + 1, \cdots, j_h^*).$$

For each h, $\Omega^* = \sum_i G_i^{*h}$ and obviously every infinite product $G_{i_1}^{*1} G_{i_2}^{*2} \cdot \cdots$ is either zero or a point. We assert that *the space* $\bar{\Omega}^*$ (Ω^* plus its ideal points defined by means $\{G_i^{*h}\}$) *is homeomorphic to the closed n-cell* $R^* + D^*$ when R^* is the region of Ω^* interior to D^*. This hardly needs proof; briefly, the radial contraction $\rho' = \rho/(1 + \rho)$ carries the whole of Ω^* into R^*, and carries the system $\{G_i^{*h}\}$ into an equivalent system for R^*. In particular, each X_i^{*h} is carried into a piece of R^* which abuts on D^* and which becomes closed when a portion of D^* is added to it. In this way we obtain a covering system for $R^* + D^* = \bar{R}^*$, and the resulting f.s.'s are in (1, 1) correspondence with the points of $\bar{R}^*$ owing to its compactness. Thus, there is a (1, 1) correspondence $\bar{R}^* \leftrightarrow \bar{\Omega}^*$, corresponding points having identical f.s.'s, and since neighborhoods can be defined in terms of the f.s.'s, the correspondence is a homeomorphism.

It is clear that the homeomorphism $\bar{R}^* \leftrightarrow \bar{\Omega}^*$ induces one between ideal sets, $\Delta^* \leftrightarrow D^*$. Let us recall the homeomorphism $D \leftrightarrow \Delta$ of 39. Let Δ' be the image in Δ of D^* in D under $D \leftrightarrow \Delta$, and let T be the homeomorphism $\Delta' \leftrightarrow \Delta^*$ obtained by combining the homeomorphisms

$$\Delta^* \leftrightarrow D^* \text{ and } \Delta' \leftrightarrow D^* .$$

Thus,

$$\alpha = T(\alpha^*) \qquad (\alpha^* \in \Delta^*)$$

implies

$$\alpha \in \Delta', \qquad \{\alpha\}_\Delta = \{\alpha^*\}_{\Delta^*} .$$

48. The following lemma offers no difficulty (cf. 41.1).

If λ^{*i} ($i = 1, 2, \cdots$) are points of Ω^* and α^* is a point of Δ^*, and if $\lambda^{*i} \to \alpha^*$, then $d(0, \lambda^{*i}) \to \infty$ and $d(0, \lambda_D^{*i}) \to a^*$ where a^* is the point of D with $\{a^*\}_{D^*} = \{\alpha^*\}_{\Delta^*}$.

49. **The space** M^*. Let M^* be an n-dimensional torus and let $\omega_1^*, \cdots, \omega_n^*$ be an independent set of generators of its fundamental group, each generator being a path beginning and ending at say O^*. Let f be a continuous single-valued mapping of M^* on M such that

$$\text{(a)}\ \ f(O^*) = 0, \qquad \text{(b)}\ f(\omega_i^*) = \omega_i \qquad (i = 1, \cdots, n, n < m).$$

If two closed paths of M^* are equivalent, then their images in M are equivalent since f is continuous, and since the image of the sum of two closed paths is the sum of their images, we have

$$\text{(c)}\ f(\lambda_1\omega_1^* + \cdots + \lambda_n\omega_n^*) = \lambda_1\omega_1 + \cdots + \lambda_n\omega_n .$$

The universal covering space of M^* is a cartesian S_n and may therefore be taken to be Ω^*. Moreover, the lattice points of Ω^*, considered as a subspace of S_m can be taken as the lattice points of Ω^* considered as the covering space of

M^*. These lattice points are of the form $\lambda = (\lambda_1^*, \cdots, \lambda_n^*, 0, \cdots, 0)$. Let Ω_λ^* be the cube

$$\lambda_i^* - \tfrac{1}{2} < x^i < \lambda_i^* + \tfrac{1}{2} \qquad (i = 1, \cdots, n)$$

$$x_i = 0 \qquad (i = n+1, \cdots, m).$$

50. If ξ^{*i}, λ^{*i} are points of Ω^*, the latter being lattice points, and if $\xi^{*i} \in \bar{\Omega}_{\lambda^{*i}}^*$, and if $\lambda^{*i} \to \alpha^*$ $(\alpha^* \in \Delta^*)$ then $\xi^{*i} \to \alpha^*$.

Proof. No difficulty (cf. 45).

51. The mapping f of M^* on M induces a mapping F of Ω^* on Λ. Let ξ^* be an arbitrary point of Ω^* and think of Ω^* as the covering space of M^*. We may then write $\xi^* = (x^*, w^*)$ when w^* is a path $\overset{\frown}{O^*x^*}$ in M^*. Let

$$\xi = F(\xi^*) = (f(x^*), f(w^*)) = \text{a point of } \Lambda.$$

F is clearly single-valued and continuous and on account of 47(c), it follows that

51.1. if λ^* is a lattice point of Ω^*, $F(\lambda^*)$ is a lattice point of Λ (hence of Ω) and its coördinates are those of λ^*.

52. $\dim F(\Omega^*) \leqq \dim f(M^*)$.

Proof. Let ξ be a point of $F(\Omega^*)$. By definition of F, we can write

$$\xi = (f(x^*), f(w^*)),\ (x^* \in M^*,\ w^* = \overset{\frown}{O^*x^*})$$
$$= (x, w) \text{ say, where } x = f(x^*) \in f(M^*).$$

Hence, by 13, $\xi \in \Lambda(f(M^*))$. Therefore $F(\Omega^*) \subseteq \Lambda\,(f(M^*))$ and hence

$$\dim F(\Omega^*) \leqq \dim \Lambda\,(f(M^*))$$
$$\leqq \dim f(M^*) \text{ by } 15.$$

53. If $\bar{\Omega}_\lambda F(\bar{\Omega}_\mu^*) \neq 0$, then

$$\bar{\Omega}_{\lambda+\tau} F(\bar{\Omega}_{\mu+\tau}^*) \neq 0 \qquad (\tau = (\tau_1, \cdots, \tau_n, 0, \cdots 0)).$$

Proof. Suppose ξ is a point of the first product. Then

$$F(\xi^*) = (f(x^*), f(w^*)), \text{ say.}$$

If w^* is replaced by $\tau_1\omega_1^* + \cdots + \tau_n\omega_n^* + w^*$, the new ξ will be in $F(\Omega_{\mu+\tau}^*)$; and since by 49(c), $f(w^*)$ is simultaneously replaced by $\tau_1\omega_1 + \cdots + \tau_n\omega_n + f(w^*)$, the new ξ also lies in $\bar{\Omega}_{\lambda+\tau}$; hence the second product is $\neq 0$.

54. There is a number B independent of λ, μ such that if $\bar{\Omega}_\lambda F(\bar{\Omega}_\mu^*) \neq 0$, then $d(\lambda, \mu) < B$.

Proof. Since $F(\bar{\Omega}_\mu^*)$ is closed and compact in Λ, it meets at most a finite

number of $\bar{\Omega}_\lambda$'s by §1 (Corollary). Combining this fact with 53, the proof now proceeds as that of 12.1.

55. We shall now extend the definition of F to the whole of $\bar{\Omega}^*$ by defining simply

$$F(\alpha^*) = T(\alpha^*) \text{ for } \alpha^* \in \Delta^* \quad \text{(see 51)}.$$

F is now single-valued over $\bar{\Omega}^*$; *we assert that F is also continuous throughout.* Since F is continuous over Ω^* and over Δ^*, we need merely show that, if $\xi^{*i} \to \alpha^*$ ($\xi^{*i} \in \Omega^*$, $\alpha^* \in \Delta^*$), then $F(\xi^{*i}) \to F(\alpha^*)$.

Proof. $F(M^*)$ is closed and compact in M; hence by 6 we can choose $\bar{n}$ so large that $f(M^*) \subseteq \bar{F}^{\bar{n}}$. We may assume Ω to have been constructed with this choice of $\bar{n}$ (see 11) and hence that $\Lambda(f(M^*)) \subseteq \Omega$ by 13. We can therefore write

$$F(\bar{\Omega}^*) \subseteq \Omega. \tag{1}$$

Now choose lattice points ξ^{*i} in Ω^* such that

$$\xi^{*i} \in \Omega^*_{\lambda^{*i}} \qquad (i = 1, 2, \cdots). \tag{2}$$

Then by 50, $\lambda^{*i} \to \alpha^*$ and by 48

$$d(0, \lambda^{*i}) \to \infty, \qquad \lambda^{*i}_D \to a^* \tag{3}$$

where $a^* \in D^*$, $\{a^*\}_{D^*} = \{\alpha^*\}_{\Delta^*}$. Now a^* is, of course, a point of D and the λ^{*i}'s are lattice points of S_m. Therefore 41.1 is applicable to the conditions (2). The result is that, if λ^i is the lattice point of Ω with same coördinates as λ^{*i}, then $\lambda^i \to \alpha$ where $\alpha \in \Delta$ and $\{\alpha\}_\Delta = \{a^*\}_D$. Thus

$$\{\alpha^*\}_{\Delta^*} = \{a^*\}_{D^*} = \{a^*\}_D = \{\alpha\}_\Delta$$

and from the relation $\{\alpha^*\}_{\Delta^*} = \{\alpha\}_\Delta$, in particular, we have $\alpha = T(\alpha^*)$ (51). Therefore $\alpha = F(\alpha^*)$. Moreover, by 51.1 $\lambda^i = F(\lambda^{*i})$ and by (2) $F(\xi^{*i}) \in F(\bar{\Omega}^*_{\lambda^{*i}})$. We can also write (on account of (1)) $F(\xi^{*i}) \in \bar{\Omega}_{\mu^i}$ for suitable μ^i and then $\bar{\Omega}_{\mu^i} F(\bar{\Omega}^*_{\lambda^{*i}}) \neq 0$. Hence $d(\lambda^{*i}, \mu^{*i}) < B$ by 54.1, which implies $d(\lambda^i, \mu^i) < B$. Therefore $\mu^i \to \alpha$ by 42 and hence $F(\xi^{*i}) \to \alpha$ by 45.

56. Denote the homeomorphism $\bar{\Omega}^* \leftrightarrow \bar{R}^*$ of 47 by Φ. Then by the preceding result, $F(\bar{\Omega}^*)$ is the single-valued continuous image of $\bar{R}^*$ under $F\Phi$; in particular the correspondence between Δ^* and D^* under $F\Phi$ is a homeomorphism.

Now $F(\bar{\Omega}^*)$ is a subset of $\bar{\Omega}$ by 55(1), and $\bar{\Omega}$ is separable by 39. Therefore the results of dimension theory can be applied to $F(\bar{\Omega}^*)$. Suppose that $\dim F(\bar{\Omega}^*) \leqq n - 1$. Then the non-singular image Δ^* of the $(n-1)$-sphere D^* would be the boundary of a (singular) chain which, as a point set in $\bar{\Omega}$, is of dimension $< n$. This situation, surely abhorrent to the intuition, is equally impossible mathematically, due to a theorem of Alexandroff.[11] We have therefore the result that, *if f satisfies the conditions (a) (b) of 49, then $\dim F(\bar{\Omega}^*) \geqq n$.*

[11] *Untersuchungen über Gestalt etc.*, Annals of Math., 30 (1928–9), p. 175, Kor. III.

II. Application to Group Manifolds

57. Let $\mathfrak{G}$ be an n-dimensional space satisying (a) and (b) of §2 and the further condition

(c) $\mathfrak{G}$ has a denumerable complete set of neighborhoods which are n-cells.[12]

Suppose now that the points of $\mathfrak{G}$ are the elements of a group with continuous multiplication and division.[13] It is well known that the fundamental group $G(\mathfrak{G})$ is then abelian.[14] We shall now show that $G(\mathfrak{G})$ *cannot have more than n linearly independent elements.* We shall assume that there are $n + 1$ linearly independent elements $u_1, \cdots, u_{n+1}$ and eventually obtain a contradiction.

Let G_{n+1} be the subgroup of $G(\mathfrak{G})$ generated by $u_1, \cdots, u_{n+1}$. Hopf[15] has introduced the concept of the "covering space of $\mathfrak{G}$ belonging to the subgroup G_{n+1}" and Schreier[16] has introduced the concept of the "covering group of $\mathfrak{G}$." A study of these concepts shows readily that they can be combined, and thus we form the "covering *group* $\bar{\mathfrak{G}}$ of $\mathfrak{G}$ belonging to G_{n+1}." The fundamental group of $\bar{\mathfrak{G}}$ is simply isomorphic with G_{n+1},[17] and can therefore be generated by a set of independent elements

$$\omega_1, \cdots, \omega_{n+1} \tag{1}$$

each of infinite order. We can show in much the same way in which we showed it for Λ, that $\bar{\mathfrak{G}}$ satisfies the conditions of §2. We may therefore take $\bar{\mathfrak{G}} = M$, $m = n + 1$. Moreover, the neighborhood condition (c) stated above will hold equally well for $\bar{\mathfrak{G}} = M$ (see footnote 12).

Let $\xi_1^*, \cdots, \xi_n^*$ $(0 \leqq \xi_i^* \leqq 2\pi)$ be the angular coordinates of the n-dimensional torus M^*. Then for the generators of the fundamental group $G(M^*)$ we may take

$$\omega_1^*, \cdots, \omega_n^*$$

where ω_i^* is obtained by allowing ξ_i^* to increase from 0 to 2π, holding the remaining ξ_i^*'s equal to 0. The ω_i^*'s are closed paths, beginning and ending at $O^* = (0, \cdots, 0)$. Now the first n ω_i's of (1) can be defined by

$$\xi_i = \tau_i(\xi_i^*) \qquad (i = 1, \cdots, n) \tag{2}$$

$$\tau_i(0) = \tau_i(2\pi) = 0$$

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[12] We add the condition (c) in order that we may draw on certain results of Hopf (*Zur Topologie der Abbildungen von Mannigfaltigkeiten*, zweite Teil, Math. Ann., 102 (1929). In the remaining footnotes we shall denote this paper by H.)

[13] We assume continuous division in order that we may refer to certain results of Schreier, —loc. cit. (The fact is, however, that our results require only continuous non-associative multiplication.) Continuous multiplication means here that the group product ab is continuous in the pair a, b.

[14] Schreier (loc. cit.).

[15] H., p. 568.

[16] Loc. cit.

[17] H., p. 572.

where the τ_i's are single-valued and continuous and 0 is the identity element of $\mathfrak{G} = M$.

Let

$$f(\xi^*) = \tau_1(\xi_1^*) \bullet \cdots \bullet \tau_n(\xi_n^*)$$

where the product on the right is the group product defined for $\mathfrak{G}$; f is a single-valued continuous mapping of M^* on M and on account of (2)

$$f(0^*) = 0 \text{ and } f(\omega_i^*) = \omega_i \qquad (i = 1, \cdots, n). \tag{3}$$

58. We now use two concepts due to Hopf. The images under f of the group elements of $G(M^*)$ constitute a subgroup of the group $G(M)$ and its index is called by Hopf the *index* of f. Now the subgroup in question is precisely that subgroup of $G_{n+1} = G(M)$ generated by $\omega_1, \cdots, \omega_n$ on account of 57 (3). Therefore *the index of f is infinite.*

The other concept is that of the *absolute grade*[18] *of f*; we need only state the following theorems, due to Hopf.

(a) If the absolute grade of f is $\neq 0$, then the index of f is finite.[19]

(b) If the absolute grade of f is zero, then f can be continuously modified to f' where the mapping f' is such that $f'(M^*)$ is nowhere dense in M.[20]

From (a) it follows that the absolute grade of f is zero. But by the use of (b) this leads to a contradiction.

For, consider the mapping f'. We may assume that $f'(O^*) = 0$, for, since M is a continuous group, we can define isotopic deformations of M into itself which will carry any given point into any other; thus f' can be further modified so as to bring $f'(O^*)$ to 0,without affecting the nowhere-denseness of $f'(M^*)$. We can now write

$$f'(O^*) = 0, \qquad f'(\omega_i^*) = \omega_i \qquad (i = 1, \cdots, n).$$

Therefore, if F' is the corresponding mapping of $\bar{\Omega}^*$ on $\Lambda + \Delta$ (see 5 and 55) we have by 56

$$\dim F'(\bar{\Omega}^*) \geqq n.$$

On the other hand, let $C = f'(M^*)$. We can write $C = C\bar{E}^1 + C\bar{E}^2 + \cdots$, the E^i's being the n-cells of M. Each $C\bar{E}^i$ is closed and nowhere dense in $\bar{E}^i$ since C is closed and nowhere dense in M; hence[21] $\dim C\bar{E}^i < n$ and hence[22] $\dim f'(M^*) = \dim C < n$; hence $\dim F'(\Omega^*) < n$ by 52. Since $F'(\Delta^*)$ is homeomorphic to an $(n-1)$-sphere, $\dim F'(\Delta^*) = n - 1$. Hence[23] $F'(\bar{\Omega}^*) < n$. This is the contradiction which concludes the proof.

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[18] H., p. 582.
[19] H., p. 584, VII.
[20] H., p. 605, XIVb and XIVc.
[21] M., p. 244.
[22] M., p. 92.
[23] M., p. 93.

59. For every n there exists an n-parameter continuous group whose fundamental group contains n linearly independent elements,—namely the n-dimensional toroidal group. It seems likely that no other n-dimensional group satisfying this condition (or possibly the stronger condition that the fundamental group have a basis of n free elements) can exist. In this connection we can only prove at present that *if* $\mathfrak{G}$ *is n-dimensional and if* $G(\mathfrak{G})$ *contains n linearly independent elements, then* $\mathfrak{G}$ *is compact.*

Proof. Let $u_1, \cdots, u_n$ be linearly independent elements of $G(\mathfrak{G})$ and let $\tilde{\mathfrak{G}}$ be the continuous covering group of $\mathfrak{G}$ belonging to the subgroup G^0 generated by $u_1, \cdots, u_n$ (see 57). Then[19] $G(\tilde{\mathfrak{G}}) \cong G^0$. Now assume that $\mathfrak{G}$ is non-compact. Then $\tilde{\mathfrak{G}}$ is also non-compact. Taking $\tilde{\mathfrak{G}}$ for M as in 57, the function f defines a mapping of M^* on $\tilde{\mathfrak{G}} = M$, and since M^* is compact, its image set is a proper subset of M. Hence by the results of Hopf[24] the absolute grade of f is zero. This, however, leads to a contradiction by the argument in 58.[25]

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[24] H., p. 585, Satz IX.

[25] It must be observed that the inequality in 56 which must be used in the argument has been derived only under the assumption that m (the number of generators of $G(M)$) exceeds n, whereas in the present case $m = n$. Obvious modifications, however, of §§ 46–56 will yield a proof of the inequality in question for the case $m = n$.

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ÜBER DAS PICK-NEVANLINNA'SCHE INTERPOLATIONSPROBLEM UND SEIN INFINITESIMALES ANALOGON

BY HERMANN WEYL

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Literatur und Fragestellung

Das in der Ueberschrift genannte Interpolationsproblem verlangt die Konstruktion der in der oberen z-Halbebene $\Im z > 0$ regulären Funktionen $w(z)$, die in diesem Gebiet der Bedingung $\Im w \geqq 0$ genügen und an vorgegebenen Stellen im Bereich: $z = \alpha_1, \alpha_2, \cdots$ vorgegebene Werte $\beta', \beta'', \cdots$ annehmen. Für eine *unendliche* Anzahl von Wertzuordnungen wurde die Frage zuerst behandelt von R. Nevanlinna, Ann. Ac. Sc. Fenn. **13**, 1919. Als der Grenzfall, in welchem alle Punkte $\alpha_1, \alpha_2, \cdots$ in den unendlichfernen Randpunkt der Halbebene zusammenrücken, ergibt sich das Stieltjes'sche Momentenproblem, das in dieser Auffassung von R. Nevanlinna in einer nachfolgenden Abhandlung, ebendort vol. **18**, 1922, bearbeitet wurde.[1] Die Rolle, welche im Interpolationsproblem das Schwarz'sche Lemma spielt, wird hier vom Julia'schen Lemma übernommen. Nevanlinna kommt, durch eine wichtige Bemerkung von Denjoy veranlasst,[2] auf beide Fragestellungen nochmals zurück in seinem Beitrag zu den Commentationes i. h. E. L. Lindelöf, 1929. Das Stieltjes'sche Momentenproblem wurde in gleicher Allgemeinheit von E. Hellinger dadurch gelöst,[3] dass er die dem Kettenbruch entsprechende Differenzengleichung in Analogie stellt mit einer, den Spektralparameter linear enthaltenden sich selbst adjungierten Differentialgleichung 2. Ordnung und darauf die Methode anwendet, die vom Verf. für solche Differentialgleichungen in seiner Habilitationsschrift, Math. Annalen **68**, 1910, pp. 221–238, entwickelt wurde. Zweck der vorliegenden Arbeit ist es, das Problem im Gebiete der Differentialgleichungen anzugeben, das in ähnlicher Weise dem Pick-Nevanlinna'schen Interpolationsproblem korrespondiert, und es—nach Abstreifung aller für das Resultat irrelevanten Spezialisierungen—auf dem zuletzt angedeuteten Wege zu bewältigen. Ich setze den Gegenstand ab ovo und eingehend auseinander, weil er uns zu neuen Spektralproblemen für Differentialgleichungen führt und weil ich an dem von Herrn Hellinger und mir eingeschlagenen Vorgehen eine wesentliche Vereinfachung anzubringen habe.

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[1] Nachdem die wesentlichen Resultate zuerst von H. Hamburger in drei Abhandlungen in den Math. Annalen **81, 82**, 1920–1921, gewonnen waren; die Hamburger'schen Arbeiten kommen jedoch für uns hier methodisch nicht in Betracht.

[2] Comptes rendus, **188**, 1929, p. 140 u. 1084.

[3] Math. Annalen **86**, 1922, p. 18.

§1. Das Interpolationsproblem auf eine Differenzengleichung reduziert. Uebergang zur korrespondierenden Differentialgleichung

Wir betrachten analytische Funktionen $w(z)$, welche in der oberen z-Halbebene, $\Im z > 0$, erklärt sind und deren Werte selber der Bedingung $\Im w \geqq 0$ genügen; sie mögen *positive Funktionen* heissen. Es sei die Aufgabe gestellt, alle solche Funktionen zu finden, welche an der Stelle $z = \alpha$ im Definitionsbereich den vorgegebenen Wert $w = \beta$ annehmen. *Die Aufgabe hat nur dann eine Lösung, wenn* $\Im\beta \geqq 0$ *gilt.* Ist $\Im\beta = 0$, so ist $w(z) = \text{const.} = \beta$ die einzige Lösung. Wenn aber $\Im\beta > 0$ ist, erhalten wir nach dem Schwarz'schen Lemma für die durch

$$\frac{w-\beta}{w-\bar{\beta}} = \frac{z-\alpha}{z-\bar{\alpha}} \cdot W(z)$$

eingeführte Funktion $W(z)$ die Ungleichung $|W| \leqq 1$. Man nehme willkürlich eine positiv imaginäre Zahl γ, $\Im\gamma > 0$, zu Hilfe und setze

$$W = \frac{w'-\gamma}{w'-\bar{\gamma}}:$$

die Bedingung $|W| \leqq 1$ wird dadurch in $\Im w' \geqq 0$ zurückverwandelt. So werden wir zu dem Ansatz geführt:

$$\frac{w-\beta}{w-\bar{\beta}} = \frac{z-\alpha}{z-\bar{\alpha}} \cdot \frac{w'-\gamma}{w'-\bar{\gamma}} \tag{1}$$

w genügt unserer Aufgabe dann und nur dann, wenn w' eine beliebige positive Funktion ist. Die Gleichung (1) lautet in aufgelöster Form

$$w = \frac{-(z\cdot\Im\bar{\beta} - \Im\bar{\beta}\alpha)\,w' + (z\cdot\Im\bar{\beta}\gamma - \Im\bar{\beta}\alpha\gamma)}{\Im\alpha\cdot w' + (z\cdot\Im\gamma - \Im\alpha\gamma)}. \tag{2}$$

Die rekursive Anwendung dieses Verfahrens liefert die Lösung des Interpolationsproblems: diejenigen positiven Funktionen $w(z)$ zu bestimmen, welche an den verschiedenen Stellen $z = \alpha_1, \alpha_2, \cdots$ des Definitionsbereiches $\Im z > 0$ vorgegebene Werte $\beta', \beta'', \cdots$ annehmen. Man beginnt mit $w_1 = w$. Der einzelne Schritt der Rekursion sieht so aus. Man habe $n-1$ Zahlen $\beta_1, \cdots, \beta_{n-1}$ gewonnen, welche den Ungleichungen genügen

$$\Im\beta_1 > 0, \cdots, \Im\beta_{n-1} > 0,$$

und eine Formel

$$w(z) = \frac{-A_{n-1}(z) + B_{n-1}(z)\cdot w_n(z)}{C_{n-1}(z) - D_{n-1}(z)\cdot w_n(z)}. \tag{3}$$

Sie soll alle positiven Funktionen $w(z)$, welche den ersten $n-1$ Bedingungen genügen,

$$w(\alpha_1) = \beta', \cdots, w(\alpha_{n-1}) = \beta^{(n-1)},$$

dadurch liefern, dass man für $w_n(z)$ eine *willkürliche* positive Funktion einsetzt. Die weitere Forderung $w(\alpha_n) = \beta^{(n)}$ setzt sich vermöge (3) in eine Gleichung $w_n(\alpha_n) = \beta_n$ um; *damit sie sich erfüllen lässt, muss* $\Im\beta_n \geqq 0$ *sein.* Ist $\Im\beta_n = 0$, so bricht das Verfahren dadurch ab, dass $w_n(z) = \text{const.} = \beta_n$ die einzige Lösung $w(z)$ liefert, welche den ersten n Werte-Zuordnungen genügt; mit $w(z)$ sind auch die weiteren Werte $\beta^{(n+1)}, \cdots$ zwangsläufig festgelegt. Ist aber $\Im\beta_n > 0$, so wählt man ein positiv imaginäres γ_n und erhält durch die Substitution

$$(4) \qquad w_n = \frac{-w_{n+1}(z\Im\bar{\beta}_n - \Im\bar{\beta}_n\alpha_n) + (z\cdot\Im\bar{\beta}_n\gamma_n - \Im\bar{\beta}_n\alpha_n\gamma_n)}{w_{n+1}\cdot\Im\alpha_n + (z\Im\gamma_n - \Im\alpha_n\gamma_n)}$$

nach dem Vorbild von (2) das nächstfolgende w_{n+1}:

$$(5) \qquad w = \frac{-A_n(z) + B_n(z)\cdot w_{n+1}}{C_n(z) - D_n(z)\cdot w_{n+1}}.$$

Daher für A_n, B_n die Rekursionsformeln:

$$(6) \qquad \begin{cases} A_n = A_{n-1}(z\Im\gamma_n - \Im\alpha_n\gamma_n) - B_{n-1}(z\Im\bar{\beta}_n\gamma_n - \Im\bar{\beta}_n\alpha_n\gamma_n), \\ B_n = -A_{n-1}\cdot\Im\alpha_n - B_{n-1}(z\Im\bar{\beta}_n - \Im\bar{\beta}_n\alpha_n), \end{cases}$$

denen auch das Paar C_n, D_n genügt. Ihr Unterschied kommt von den Anfangswerten her:

$$(7) \qquad A_0 = 0, B_0 = 1; \qquad C_0 = 1, D_0 = 0.$$

z sei ein fester Wert in der oberen Halbebene. Lassen wir in der Formel (5) die Grösse w_{n+1} (unabhängig von z) über den Bereich $\Im w_{n+1} \geqq 0$ frei variieren, so durchläuft w eine gewisse Kreisscheibe $\mathfrak{k}_n = \mathfrak{k}_n(z)$. Nach der Herleitung sind diese Kreise ineinander eingeschachtelt und liegen alle in der oberen Halbebene $\mathfrak{k}_0$: $\mathfrak{k}_0 \supset \mathfrak{k}_1 \supset \mathfrak{k}_2 \supset \cdots$. Unter der Voraussetzung $\Im\beta_\nu > 0$ $(\nu = 1, 2, \cdots, n)$ sind die Gleichungen $w(\alpha_\nu) = \beta^{(\nu)} (\nu = 1, \cdots, n)$ für die positive Funktion $w(z)$ der Forderung äquivalent, dass *der Wert* $w(z)$ *für alle positiv imaginären* z *in der Kreisscheibe* $\mathfrak{k}_n(z)$ *gelegen ist.* Denn genügt $w(z)$ dieser Forderung und führt man durch (5) eine analytische Funktion $w_{n+1}(z)$ ein, so erfüllt diese die Bedingung $\Im w_{n+1} \geqq 0$.

Es entsteht nunmehr die *transzendente Frage,* wie weit die positive Funktion $w(z)$ bestimmt ist,wenn die Stellen $z = \alpha_1, \alpha_2, \cdots$, an denen $w(z)$ vorgegebene Werte $\beta', \beta'', \cdots$ annehmen soll, *in unendlicher Anzahl* verhanden sind. Es ist vorausgesetzt, dass alle Ungleichungen $\Im\beta_\nu > 0$ $(\nu = 1, 2, \cdots,$ in inf.) erfüllt sind. Das Hauptresultat der Nevanlinna'schen Theorie ist dies: *Analog dem, was wir bei endlichem* n *gesehen haben, gibt es zwei Fälle, den "Grenzpunkt"- und den "Grenzkreis"-Fall. Im ersten gibt es eine und nur eine Lösung unseres Problems; im zweiten erhält man die allgemeinste Lösung in der Form*

$$(8) \qquad w(z) = \frac{-A(z) + B(z)\cdot w_\infty(z)}{C(z) - D(z)\cdot w_\infty(z)},$$

wo A, B, C, D wohlbestimmte analytische Funktionen von z in der oberen Halbebene sind, während für $w_\infty(z)$ eine willkürliche positive Funktion eintreten kann.

Die Fallunterscheidung resultiert sofort aus der Betrachtung der Kreise $\mathfrak{k}_n$: infolge der Einschachtelung ziehen sie sich entweder auf einen Punkt (Grenzpunkt) oder auf einen Kreis (Grenzkreis) $\mathfrak{k}_\infty$ zusammen. Im ersten Fall muss der Wert von $w(z)$ für den betrachteten Argumentwert z notwendig dieser Grenzpunkt sein. Es entsteht die Aufgabe, einzusehen, *dass die Unterscheidung von Grenzpunkt und Grenzkreis unabhängig ist von dem betrachteten Werte von z.*

Wiederum sei die positiv imaginäre Zahl z fest gewählt. Um die Differenzengleichungen (6) in Parallele stellen zu können zu Differentialgleichungen, ist es bequem, dafür zu sorgen, dass in diesen Rekursionsformeln die Koeffizienten in der Hauptdiagonale auf der rechten Seite übereinstimmen. Das geschieht durch den Ansatz $\gamma_n = -\bar{\beta}_n$, der mit der Forderung $\Im\gamma_n > 0$ in Einklang steht. Der in A_n, B_n, C_n, D_n verbleibende willkürliche gemeinsame Faktor war in jenen Formeln so normiert, dass sich diese Grössen als Polynome in z vom Grade n ergaben. Für die Differentialgleichung ist das ohne Belang; hier ist es vielmehr zweckmässig, dafür zu sorgen, dass die Koeffizienten in der Hauptdiagonale $= 1$ werden. Man setze darum

$$(9)\qquad A_n = \prod_{\nu=1}^{n} (z\Im\gamma_\nu - \Im\alpha_\nu\gamma_\nu)\cdot f(n)\,, \qquad B_n = \prod_{\nu=1}^{n} (z\Im\gamma_\nu - \Im\alpha_\nu\gamma_\nu)\cdot g(n)\,.$$

Dann erhalten wir statt (6):

$$(10)\qquad \begin{cases} f(n) = f(n-1) + g(n-1)\cdot\dfrac{za_n'' - b_n''}{za_n' - b_n'}\,, \\ g(n) = g(n-1) + f(n-1)\cdot\dfrac{za_n - b_n}{za_n' - b_n'}\,. \end{cases}$$

Hier ist

$$(11)\qquad \begin{cases} a = \Im 1\,, & a' = \Im\gamma\,, & a'' = \Im\gamma^2\,; \\ b = \Im\alpha\,, & b' = \Im\alpha\gamma\,, & b'' = \Im\alpha\gamma^2 \end{cases}$$

gesetzt; überall ist der Index n hinzuzufügen.

Wie aus dem Folgenden hervorgehen wird, kommt es auf die besonderen Ausdrücke (11) nicht an, es ist—für die korrespondierende Differentialgleichung —allein wesentlich, dass *die beiden Determinanten*

$$p = a'b - b'a, \qquad q = a''b' - b''a'$$

positiv sind. Mit den Ausdrücken (11) ist in der Tat $p = \Im\alpha\cdot\Im\gamma$. q ist der imaginäre Teil von

$$\alpha\gamma\cdot\Im\gamma^2 - \alpha\gamma^2\cdot\Im\gamma = \frac{1}{2i}\,[\alpha\gamma(\gamma^2 - \bar{\gamma}^2) - \alpha\gamma^2(\gamma - \bar{\gamma})]$$

$$= \frac{1}{2i}\,(\alpha\gamma^2\bar{\gamma} - \alpha\gamma\bar{\gamma}^2) = \alpha\gamma\bar{\gamma}\cdot\frac{\gamma - \bar{\gamma}}{2i} = \alpha|\,\gamma\,|^2\cdot\Im\gamma\,,$$

also

$$q = \Im\alpha \cdot \Im\gamma \cdot |\gamma|^2 > 0 .$$

In der Gestalt (10) ist der Uebergang zur Differentialgleichung trivial. Man ersetze die diskrete Variable n durch eine kontinuierliche s, die das Intervall von 0 bis $+\infty$ durchläuft. z ist der gewöhnlich mit λ bezeichnete Spektralparameter, für den nur positiv imaginäre Werte zugelassen sind.

$$a(s),\ a'(s),\ a''(s); \qquad\qquad b(s),\ b'(s),\ b''(s)$$

sind stetige reelle Funktionen von s, welche den beiden Ungleichungen genügen

$$(12) \qquad p(s) = a'b - b'a > 0, \qquad\qquad q(s) = a''b' - b''a' > 0.$$

Wir behandeln *das System der zwei Differentialgleichungen 1. Ordnung für die unbekannten Funktionen* $f(s)$, $g(s)$:

$$(13) \qquad \begin{cases} \dfrac{df}{ds} = g(s) \cdot \dfrac{za''(s) - b''(s)}{za'(s) - b'(s)}, \\ \dfrac{dg}{ds} = f(s) \cdot \dfrac{za(s) - b(s)}{za'(s) - b'(s)}. \end{cases}$$

Der Spektralparameter z tritt hier *linear gebrochen* auf, während er in die vielbehandelten klassischen Eigenwertprobleme ganz-linear eingeht. Diejenige Lösung, welche die Anfangswerte $f = 0$, $g = 1$ für $s = 0$ besitzt, werde wie in meiner oben zitierten Habilitationsschrift durch den Buchstaben ϑ gekennzeichnet, genauer $(f_\vartheta, g_\vartheta)$; diejenige mit den Anfangswerten $f = 1$, $g = 0$ heisse $\eta = (f_\eta, g_\eta)$.

In den entsprechenden Differenzengleichungen (10) führt $(f_\vartheta, g_\vartheta)$ mittels (9) zu (A, B) und (f_η, g_η) zu (C, D). Die Gleichung (5) lautet darum

$$(14) \qquad w = \frac{-f_\vartheta(n) + g_\vartheta(n) \cdot h}{f_\eta(n) - g_\eta(n) \cdot h}.$$

Hier ist h anstelle von w_{n+1} geschrieben. w durchläuft die Peripherie des Kreises $\mathfrak{k}_n$, wenn h die reelle Achse (incl. ∞) beschreibt. (14) lässt sich in die Form setzen:

$$(15) \qquad f(n) - h \cdot g(n) = 0,$$

wo $\omega = (f, g)$ die aus den beiden Partikularlösungen ϑ und η durch lineare Kombination entstehende Lösung

$$f(n) = f_\vartheta(n) + w \cdot f_\eta(n), \qquad\qquad g(n) = g_\vartheta(n) + w \cdot g_\eta(n)$$

ist. w liegt also dann und nur dann auf der Peripherie von $\mathfrak{k}_n$, wenn die zugehörige Lösung ω an der Stelle n einer reellen Randbedingung (15) genügt. In dieser Gestalt übertrage ich die Definition auf die Differentialgleichung: w sei eine willkürliche Konstante, und man betrachte die Lösung $\omega = (f, g)$:

$$(16) \qquad f(s) = f_\vartheta(s) + w \cdot f_\eta(s), \qquad\qquad g(s) = g_\vartheta(s) + w \cdot g_\eta(s)$$

der Differentialgleichungen (13) im endlichen Intervall $0 \leqq s \leqq l$. Sie genügt an dem Ende $s = l$ dann und nur dann einer reellen Randbedingung, wenn w auf einem gewissen Kreise $(\mathfrak{k}_l)$ liegt. In solcher Weise hatte ich in meiner Habilitationsschrift die Kreise $(\mathfrak{k}_l)$ eingeführt. Der Parameter w ist dort mit l bezeichnet.[4]

§2. Green'sche Formel. Die Kreise $\mathfrak{k}_l$

Wir benötigen die *"Green'sche Formel"* für eine Lösung $\omega = (f, g)$ der Gleichungen (13) und eine Lösung $\omega^* = (f^*, g^*)$ der entsprechenden Gleichungen mit dem Parameterwert z^* statt z. Für die Ableitung der Determinante

$$(\omega\omega^*) = fg^* - gf^*$$

nach s finden wir mittels einer leichten Rechnung, indem wir für die Derivierten $\frac{df}{ds}, \cdots$ ihre Werte aus den Differentialgleichungen einsetzen, den Ausdruck:

$$-(z - z^*) \cdot \frac{D(s; \omega\omega^*)}{(za' - b')(z^*a' - b')},$$

wo

$$D(s; \omega\omega^*) = p(s)f(s)f^*(s) + q(s)g(s)g^*(s) \tag{17}$$

ist. Darum lautet die gewünschte Formel:

$$(z - z^*) \cdot \int_0^l \frac{D(s; \omega\omega^*)\, ds}{(za'(s) - b'(s))(z^*a'(s) - b'(s))} = -(\omega\omega^*)_0^l. \tag{18}$$

In dreierlei Weise findet sie Anwendung: 1) für $z^* = z$, 2) wenn z und z^* zwei positiv imaginäre Werte sind, 3) für $z^* = \bar{z}$. Im Falle 1) kommt

$$(\omega\omega^*)_0^l = 0,$$

d.h. zwei Lösungen ω und ω^* der Differentialgleichungen (13) haben eine konstante Determinante $(\omega\omega^*)$. Insbesondere ist $(\eta\vartheta)_s = (\eta\vartheta)_0 = 1$, und wenn ω die Lösung $\vartheta(s) + w \cdot \eta(s)$ bezeichnet, gilt auch $(\eta\omega) = 1$. Anwendung 3) liefert die für jede Lösung ω unserer Gleichungen giltige Beziehung

$$\Im z \cdot \int_0^l \frac{D(s; \omega\bar{\omega})}{|za'(s) - b'(s)|^2}\, ds = (\omega_1\omega_2)_0^l. \tag{19}$$

Hier sind wie im Folgenden stets Real- und Imaginärteil einer Grösse durch den angehängten Index 1 und 2 gekennzeichnet. Nunmehr erkennt man die Bedeutung der Voraussetzungen (12): *sie garantieren, dass der "Dirichlet-*

[4] Die unsymmetrische Behandlung der beiden Partikularlösungen ϑ, η ist hier nur scheinbar. Statt (16) könnte man die allgemeinste Lösung $\omega(s) = w_\vartheta \cdot \vartheta(s) + w_\eta \cdot \eta(s)$ betrachten. Da die Kreisscheibe $\mathfrak{k}_l$ aber sowohl den Punkt 0 wie den Punkt ∞ ausschliesst, kann man statt der beiden homogenen Parameter w_ϑ und w_η entweder den inhomogenen $w = w_\eta/w_\vartheta$ oder w_ϑ/w_η benutzen. Wir entschieden uns für das Erste.

Ausdruck" $D(s;\omega\bar{\omega})$ positiv-definit ist.—Der Fall 2) der Formel (18) wird erst in §3 benutzt werden.

Dass diejenigen Werte w, für welche die Lösung ω, Gl.(16), am Ende l des Intervalls $0 \leqq s \leqq l$ einer reellen Randbedingung genügt:

$$f(l) - h \cdot g(l) = 0, \qquad h \text{ reell}, \tag{20}$$

auf einem Kreise $(\mathfrak{k}_l)$ liegen, geht natürlich aus der Gleichung (20) oder

$$w = \frac{-f_\vartheta(l) + hg_\vartheta(l)}{f_\eta(l) - hg_\eta(l)}$$

unmittelbar hervor. Doch kann man dasselbe Resultat auch auf dem folgenden Wege gewinnen, der zugleich zu einer einfachen Berechnung des Durchmessers führt. Bei festem reellen h ist $f(l) - h \cdot g(l)$ eine ganze lineare Funktion von w. Die Forderung, dass erstens der Real- und zweitens der Imaginär-Teil davon verschwindet, gibt daher zwei zueinander senkrechte Geraden $(h)_1$ und $(h)_2$ in der w-Ebene, die sich in dem gesuchten Punkte w schneiden. Die Linie $(h)_1$ geht *für alle* h durch denjenigen Punkt $w = w^0$ hindurch, der durch die beiden Gleichungen $f_1(l) = 0$, $g_1(l) = 0$ gegeben ist; die Linie $(h)_2$ durch denjenigen Punkt $w = w_0$, der sich aus $f_2(l) = g_2(l) = 0$ ergibt. Wenn h variiert, bewegt sich also w auf einem Kreise über dem Durchmesser $w^0 w_0$ (geometrischer Ort des Scheitels eines rechten Winkels, dessen Schenkel durch zwei feste Punkte w^0 und w_0 gehen). w^0 kann aus den beiden Gleichungen bestimmt werden

$$(\eta_1 \omega_1)_l = 0, \qquad (\eta_2 \omega_1)_l = 0.$$

Wegen $(\eta\omega) = 1$ oder

$$(\eta_1 \omega_1) - (\eta_2 \omega_2) = 1, \qquad (\eta_1 \omega_2) + (\eta_2 \omega_1) = 0$$

kann hier die zweite Gleichung durch $(\eta_1 \omega_2) = 0$ ersetzt werden, sodass sich

$$(\eta_1 \omega) = 0, \qquad (\eta_1 \vartheta) + w(\eta_1 \eta) = 0$$

ergibt oder

$$(\eta_1 \vartheta)_l + iw^0 (\eta_1 \eta_2)_l = 0.$$

Ebenso folgt für w_0 die Gleichung $(\eta_2 \omega) = 0$ oder

$$(\eta_2 \vartheta)_l - w_0 (\eta_1 \eta_2)_l = 0.$$

Daraus kommt für die Differenz $w^0 - w_0$:

$$(\eta\vartheta)_l + i(w^0 - w_0)(\eta_1 \eta_2)_l = 0, \qquad w^0 - w_0 = \frac{i}{(\eta_1 \eta_2)_l}.$$

Weil $(\eta_1 \eta_2)_0 = 0$ ist, erhält man für den Nenner aus (19) den Ausdruck

$$(\eta_1 \eta_2)_l = z_2 \cdot \int_0^l \frac{D(s; \eta\eta)}{|za'(s) - b'(s)|^2}\, ds.$$

$w_0 w^0$ ist folglich der vertikale Durchmesser des Kreises $(\mathfrak{k}_l)$, w^0 der höchste, w_0 der tiefste Punkt und sein reziproker Durchmesser

$$1/d_l = z_2 \cdot \int_0^l \frac{D(s;\eta\bar{\eta})}{|za'(s) - b'(s)|^2}\, ds \tag{21}$$

Die Formel lässt unmittelbar erkennen, dass d_l mit wachsendem l abnimmt.

Die Tatsache, dass $\omega(s) = \vartheta(s) + w \cdot \eta(s)$ an der Stelle $s = l$ einer reellen Randbedingung genügen soll, lässt sich auch in der Gleichung $(\omega_1\omega_2)_l = 0$ aussprechen, die auf Grund von (19) der Relation

$$z_2 \cdot \int_0^l \frac{D(s;\omega\bar{\omega})}{|za'(s) - b'(s)|^2}\, ds = -(\omega_1\,\omega_2)_0$$

äquivalent ist. Man sieht sofort, dass die rechte Seite $= w_2 = \Im w$ ist. Wir erhalten somit als Gleichung des Kreises $(\mathfrak{k}_l)$:

$$z_2 \cdot \int_0^l \frac{D(s;\omega\bar{\omega})}{|za'(s) - b'(s)|^2}\, ds = w_2 \qquad (\text{für } \omega = \vartheta + w \cdot \eta).$$

Da der Koeffizient von $w\bar{w}$ hierin positiv ist, wird die vom Kreise $(\mathfrak{k}_l)$ begrenzte Kreisscheibe $\mathfrak{k}_l$ durch die Ungleichung beschrieben:

$$\Im z \cdot \int_0^l \frac{D(s;\omega\bar{\omega})\, ds}{|za'(s) - b'(s)|^2} \leqq \Im w \tag{22}$$

(mit dem Zeichen $\leqq$ und nicht etwa mit dem umgekehrten Zeichen $\geqq$). Daraus ergibt sich nun sogleich die *Tatsache der Einschachtelung: Ist $l' > l$, so liegt der Kreis $\mathfrak{k}_{l'}$ in $\mathfrak{k}_l$.* Denn ist $z_2 \cdot \int_0^{l'} \leqq w_2$, so ist a fortiori $z_2 \cdot \int_0^{l} \leqq w_2$, weil der Integrand in (22) positiv ist.

Die Kreisscheibe $\mathfrak{k}_l$ zieht sich mit unbegrenzt wachsendem l entweder auf *einen Grenzpunkt oder einen Grenzkreis* $\mathfrak{k}_\infty$ zusammen, der in allen Kreisen $\mathfrak{k}_l$ enthalten ist. Für einen Wert w, der zu $\mathfrak{k}_\infty$ gehört, gilt darum

$$z_2 \cdot \int_0^l \frac{D(s;\omega\bar{\omega})\, ds}{|za'(s) - b'(s)|^2} \leqq w_2$$

identisch in l, und darum konvergiert

$$\int_0^\infty \frac{D(s;\omega\bar{\omega})\, ds}{|za'(s) - b'(s)|^2} \text{ und ist } \leqq w_2/z_2\,. \tag{23}$$

Die Gleichungen (13) *haben demnach stets mindestens eine im Sinne des Ausdrucks* (23) *von* 0 *bis* ∞ *„quadratisch integrierbare" nicht-verschwindende Lösung* ω. *Die Formel* (21) *lehrt: der Grenzpunkt- oder der Grenzkreis-Fall liegt vor, je nachdem*

$$\int_0^\infty \frac{D(s;\eta\bar{\eta})\, ds}{|za'(s) - b'(s)|^2} \quad \textit{konvergiert oder divergiert.} \tag{24}$$

Dass in diesem Kriterium gerade die Lösung η bevorzugt wird, ist zufällig. Im Grenzkreisfall ist ausser $\eta(s)$ auch $\vartheta(s) + w \cdot \eta(s)$ quadratisch integrierbar, wenn w irgend ein Punkt im Grenzkreis ist; folglich auch $\vartheta(s)$ und damit jede Lösung überhaupt. Die Fallunterscheidung lässt sich darum auch so aussprechen: im Grenzkreisfall sind alle Lösungen quadratisch integrierbar, im Grenzpunktfall nur die Multipla einer bestimmten Lösung.

Den Anschluss an die Fragestellung, von welcher wir in §1 ausgingen, vollziehen wir durch die folgende Bemerkung. Drückt sich die in der oberen Halbebene $\Im z > 0$ definierte analytische und "positive" Funktion $w(z)$ mittels einer ebensolchen Funktion $w_s(z)$ in der Gestalt aus

$$w(z) = \frac{-f_\vartheta(z;s) + g_\vartheta(z;s) w_s(z)}{f_\eta(z;s) - g_\eta(z;s)\, w_s(z)},$$

so wollen wir sagen: $w(z)$ habe die *Eigenschaft* E_s. Sie ist der Forderung äquivalent, dass der Wert $w(z)$ für jeden positiv imaginären Wert von z in dem Kreise $\mathfrak{k}_s(z)$ liegt. Der Einschachtelungssatz besagt, dass durch E_t die Funktion $w(z)$ stärker eingeschränkt wird als durch E_s, wenn $t > s$ ist. (Man erinnere sich, dass im Differenzenproblem E_n aus E_{n-1} entstand durch Hinzufügung der neuen Bedingung $w(\alpha_n) = \beta^{(n)}$.) Die Eigenschaft $E_\infty = \lim_{s \to \infty} E_s$ besteht somit darin, dass *alle* E_s erfüllt sind. *Sie ist der Forderung gleichwertig, dass $w(z)$ für alle Werte von z in $\mathfrak{k}_\infty(z)$ liegt.*

§3. Vergleich verschiedener Werte des Spektralparameters

z, z^* seien zwei positiv imaginäre Werte des Spektrumsparameters; wir wollen zeigen: *Tritt für z der Grenzkreisfall ein, so auch für z^*.*—Unter Vertauschung von z und z^* folgt daraus: *Tritt für z der Grenzpunktfall ein, so auch für z^*.* Die Fallunterscheidung Grenzpunkt—Grenzkreis ist demnach unabhängig von dem Wert des Spektralparameters.

Das Kriterium für Grenzkreis ist die Konvergenz des Integrals (24). Beim Uebergang von z zu z^* muss man beachten, dass in dem Integral der Nenner

$$|za'(s) - b'(s)|^2 \quad \text{in} \quad |z^*a'(s) - b'(s)|^2$$

abgeändert wird. Für die Frage der Konvergenz des Integrals ist dies jedoch irrelevant auf Grund des folgenden elementaren

HILFSSATZES: Sind z, z^* zwei gegebene positiv imaginäre Zahlen, so liegt

$$\left| \frac{za - b}{z^*a - b} \right|$$

für alle reellen Zahlenpaare $(a, b) \neq (0, 0)$ zwischen festen positiven Grenzen.

Wir brauchen lediglich die obere Grenze. Sie ist

$$\frac{|z - \bar{z}^*| + |z - z^*|}{|z^* - \bar{z}^*|} = \frac{|z - \bar{z}|}{|z^* - \bar{z}| - |z^* - z|}. \tag{25}$$

Man mache sich die einfache geometrische Bedeutung dieser Ausdrücke an dem Trapez mit den Ecken $z, \bar{z}; z^*, \bar{z}^*$ klar!

Zum Beweise setze man

$$\frac{c-\bar{z}}{c-z} = \zeta .$$

In dem abzuschätzenden Quotienten

$$Q = \left| \frac{c-z^*}{c-z} \right|$$

führe man statt der Variablen c, welche die reelle Achse durchläuft, die Variable ζ auf dem Einheitskreis ein: $|\zeta| = 1$. Man erhält dann

$$Q = \left| \frac{(z^*-\bar{z}) - (z^*-z)\zeta}{z-\bar{z}} \right|$$

und

$$\frac{|z^*-\bar{z}| - |z^*-z|}{|z-\bar{z}|} \quad \text{bezw.} \quad \frac{|z^*-\bar{z}| + |z^*-z|}{|z-\bar{z}|}$$

als untere und obere Grenze. Von den beiden Ausdrücken (25) ist der zweite das Reziproke der unteren Grenze, der erste entsteht aus der oberen Grenze durch Vertauschung von z und z^*.

Das zum Parameterwert z^* gehörige $\eta^* = (f_\eta^*, g_\eta^*)$ können wir aus dem zu z gehörigen η und ϑ mittels einer *Integralgleichung* bestimmen. Wir haben nämlich nach (18)

$$(26)\quad \begin{cases} (f_\eta^* g_\eta - g_\eta^* f_\eta)_s = (z-z^*) \cdot \int_0^s \dfrac{D(t;\eta\eta^*)\,dt}{(za'(t)-b'(t))\,(z^*a'(t)-b'(t))}, \\ (f_\eta^* g_\vartheta - g_\eta^* f_\vartheta)_s = 1 + (z-z^*) \cdot \int_0^s \dfrac{D(t;\vartheta\eta^*)\,dt}{(za'(t)-b'(t))\,(z^*a'(t)-b'(t))}. \end{cases}$$

Daraus durch Auflösung nach $f_\eta^*(s)$ und $g_\eta^*(s)$:

$$(27)\quad \begin{cases} f_\eta^*(s) = f_\eta(s) + (z-z^*) \cdot \int_0^s \dfrac{f_\eta(s)D(t;\vartheta\eta^*) - f_\vartheta(s)D(t;\eta\eta^*)}{(za'(t)-b'(t))(z^*a'(t)-b'(t))}\,dt, \\ g_\eta^*(s) = g_\eta(s) + (z-z^*) \cdot \int_0^s \dfrac{g_\eta(s)D(t;\vartheta\eta^*) - g_\vartheta(s)D(t;\eta\eta^*)}{(za'(t)-b'(t))(z^*a'(t)-b'(t))}\,dt. \end{cases}$$

Diese Gleichungen für (f_η^*, g_η^*) subsumieren sich offenbar unter das Schema einer Volterra'schen Integralgleichung

$$\varphi(s) = f(s) + \int_0^s K(s,t) \cdot \varphi(t)\, \rho(t)\, dt .$$

$\rho > 0$ und f sind darin gegebene Funktionen. Wir setzen voraus, dass f und der Kern K quadratisch integrierbar sind, in dem Sinne, dass die Integrale

$$\int_0^\infty |\, f(s)\,|^2\, \rho(s)\, ds\,, \qquad \int_0^\infty \int_0^\infty |\, K(s,t)\,|^2\, \rho(s)\, \rho(t)\, dt\, ds$$

endliche Werte haben. Die Quadratwurzel aus dem ersten Integral werde zur Abkürzung mit $\| f \|$ beziechnet, und wenn als Grenzen a und b statt 0 und ∞ genommen werden, mit $\| f \|_a^b$. Unsere Behauptung ist, dass *alsdann auch φ im gleichen Sinne quadratisch integrierbar ist.* Beweis: Man multipliziere die Integralgleichung mit $\rho(s)\bar{\varphi}(s)$, integriere im Intervall $a \leqq s \leqq b$ und wende dann auf die beiden Summanden rechts die Schwarz'sche Ungleichung an. So kommt, wenn

$$\int_a^\infty \int_0^\infty |\, K(s,t)\,|^2\, \rho(s)\rho(t)\, dt\, ds = k_a^2$$

gesetzt wird, nach Kürzung durch den Faktor $\| \varphi \|_a^b$:

$$\| \varphi \|_a^b \leqq \| f \|_a^\infty + k_a \| \varphi \|_0^b\,,$$

und wegen $\| \varphi \|_0^b \leqq \| \varphi \|_0^a + \| \varphi \|_a^b$:

$$(1 - k_a) \| \varphi \|_a^b \leqq \| f \|_a^\infty + k_a \| \varphi \|_0^a\,.$$

k_a strebt mit $a \to \infty$ gegen 0; a werde so angenommen, dass bereits $k_a < 1$ ist. Dann enthält die letzte Ungleichung eine von b unabhängige obere Schranke für $\| \varphi \|_a^b$. Daraus folgt die Konvergenz von $\| \varphi \|_a^\infty$ samt der expliziten Abschätzung

$$(1 - k_a) \| \varphi \|_a^\infty \leqq \| f \|_a^\infty + k_a \| \varphi \|_0^a\,. \tag{28}$$

Es ist klar, wie dieses Schema auf (27) zur Anwendung kommt. Statt $\rho(s)\varphi(s)\bar{\varphi}(s)$ hat man hier zu bilden

$$\frac{p(s)f_\eta^*(s)\bar{f}_\eta^*(s) + q(s)g_\eta^*(s)\bar{g}_\eta^*(s)}{|\, z^*a'(s) - b'(s)\,|^2} = \frac{D(s;\eta^*\bar{\eta}^*)}{|\, z^*a'(s) - b'(s)\,|^2}$$

und bekommt so

$$\begin{aligned} D(s;\eta^*\eta^*) &= D(s;\eta\bar{\eta}^*) \\ &\quad + (z - z^*)\cdot \int_0^s \frac{D(s;\eta\bar{\eta}^*)D(t;\vartheta\eta^*) - D(s;\vartheta\bar{\eta}^*)D(t;\eta\eta^*)}{(za'(t) - b'(t))(z^*a'(t) - b'(t))}\, dt\,. \end{aligned} \tag{29}$$

Der immer wieder auftretende Nenner $za'(s) - b'(s)$ werde zur Abkürzung mit $(z;s)$, sein absoluter Betrag mit $|\, z;s\,|$ bezeichnet. Auf der rechten Seite wird man für beide $D(t)$ davon Gebrauch machen, dass zufolge der Schwarz'schen Ungleichung

$$|\, D(t;\vartheta\eta^*)\,|^2 \leqq D(t;\vartheta\bar{\vartheta})\cdot D(t;\eta^*\bar{\eta}^*)\,,$$

$$\int_0^s \frac{|\, D(t;\vartheta\eta^*)\,|\, dt}{|\, z;t\,|\cdot|\, z^*;t\,|} \leqq \| \vartheta \|_0^L \cdot \| \eta^* \|_0^L$$

ist, falls $s \leqq L$. Hier wurde gesetzt

$$\int_l^L \frac{D(t;\vartheta\bar{\vartheta})\,dt}{|z;t|^2} = (\|\vartheta\|_l^L)^2, \qquad \int_l^L \frac{D(t;\eta^*\eta^*)}{|z^*;t|^2}\,dt = (\|\eta^*\|_l^L)^2.$$

Für z liegt nach Voraussetzung der Grenzkreisfall vor; darum konvergieren die Integrale

$$\|\eta\|_0^\infty, \qquad \|\vartheta\|_0^\infty.$$

Dividiert man (29) durch $|z^*; s|^2$ und integriert nach s zwischen l und L, so treten nicht sie nach Anwendung der Schwarz'schen Ungleichung auf der rechten Seite auf, sondern der Integrand erscheint multipliziert mit einem "störenden Faktor"

$$\left|\frac{za'(s) - b'(s)}{z^*a'(s) - b'(s)}\right|^2. \tag{30}$$

Er ist aber nach dem Hilfssatz ohne Einfluss auf die Konvergenz, da er unterhalb einer von s unabhängigen Schranke m liegt. Man erhält nach dem Schema (28):

$$(1 - k_l)\,\|\eta^*\|_l^\infty \leqq m\cdot\|\eta\|_l^\infty + k_l\cdot\|\eta^*\|_0^l, \tag{31}$$

wo

$$k_l = m\,|z - z^*|\,(\|\eta\|_0^\infty\,\|\vartheta\|_l^\infty + \|\vartheta\|_0^\infty\,\|\eta\|_l^\infty) \tag{32}$$

ist und l so gewählt werden muss, dass $k_l < 1$ ausfällt.

Uebrigens kann in dieser ganzen Ueberlegung ϑ ersetzt werden durch ein $\omega = \vartheta + w\cdot\eta$, das einem Punkt w im Grenzkreis entspricht. Dann ist

$$\|\omega\|_0^\infty \leqq \sqrt{w_2/z_2}.$$

In unnatürlicher Weise hatte ich in meiner Habilitationsschrift zur Bestimmung von η^* die Randwert- statt der Anfangswert-Aufgabe benutzt und sie überdies mit Hilfe der Eigenfunktionen gelöst. Dieser Weg ist offenbar hier ungangbar; denn zu den Differentialgleichungen (13) gehört kein vernünftiges Eigenwertproblem, weil für reelle z der Nenner $(z; s)$ null werden kann. Herr Hellinger hatte die Darstellung durch Eigenfunktionen ersetzt durch die quellenmässige Darstellung mittels der Green'schen Funktion, blieb aber auch noch an der Randwertaufgabe hängen. In einer peinlichen Fallunterscheidung verrät sich das Künstliche des Vorgehens. Der hier gegebene elementare Beweis ermöglicht die explizite Abschätzung.

Dies ist nicht ohne Belang. *Im Grenzkreisfall* werden die Punkte w_s auf der Peripherie des Kreises $\mathfrak{k}_l(z)$ durch die Gleichung geliefert:

$$w_s = \frac{-f_\vartheta + hg_\vartheta}{f_\eta - hg_\eta} \qquad (s = l;\ h \text{ reell}),$$

die Punkte auf der Peripherie des Kreises $\mathfrak{k}_l(z^*)$ durch die entsprechende Gleichung

$$w_{z^*} = \frac{-f^*_\vartheta + hg^*_\vartheta}{f^*_\eta - hg^*_\eta}.$$

Ordnet man die Punkte der beiden Kreise projektiv einander zu durch gleiche Werte von h, so erhält man

$$w_{z^*} = -\frac{(\vartheta^*\vartheta)_l + (\vartheta^*\eta)_l \cdot w_z}{(\eta^*\vartheta)_l + (\eta^*\eta)_l \cdot w_z}.$$

Die hier auftretenden Determinanten $(\eta^*\eta)_l$ und $(\eta^*\vartheta)_l$ sind in (26) angegeben; für $(\vartheta^*\eta)_l$, $(\vartheta^*\vartheta)_l$ gelten entsprechende Ausdrücke. Es geht daraus hervor, dass sie mit $l \to \infty$ gegen bestimmte Grenzwerte streben. Nehmen wir für z einen festen Wert z_0, z.B. $z_0 = i$, und bezeichnen das im Gebiet $\Im z^* > 0$ variable z^* dann mit z, so ist, wie die explizite Abschätzung (31), (32) lehrt, die Konvergenz sogar *gleichmässig in* z. Darum ergeben sich im Limes *analytische Funktionen von* z. Sie sind die Koeffizienten derjenigen linearen Substitution, welche die Kreisscheibe $\mathfrak{k}_\infty(z_0)$ in die Kreisscheibe $\mathfrak{k}_\infty(z)$ überführt. Bildet man die obere Halbebene durch eine lineare Substitution auf den Kreis $\mathfrak{k}_\infty(z_0)$ ab, so bekommt man eine in z analytische Substitution:

$$w = \frac{-A(z) + B(z)\, w_\infty}{C(z) - D(z)\, w_\infty},$$

welche für jedes z die obere Halbebene $\Im w_\infty \geqq 0$ auf den Kreis $\mathfrak{k}_\infty(z)$ abbildet. *Eine analytische Funktion* $w(z)$ *von der Eigenschaft* E_∞ *muss darum die Gestalt* (8) *besitzen, wo* $w_\infty(z)$ *eine analytische und "positive" Funktion ist.* Umgekehrt hat jede analytische Funktion $w(z)$ von dieser Gestalt die Eigenschaft E_∞.

Im Grenzpunktfall muss der Wert der gesuchten Funktion von der Eigenschaft E_∞ notwendig *der Grenzpunkt selbst* $w^* = w(z)$ sein. Dass er *analytisch* von z abhängt, würde man am einfachsten erkennen, wenn aus unserm Beweise hervorginge, dass der Durchmesser $d_l(z)$ gleichmässig in z mit $l \to \infty$ gegen 0 strebt. Aber dazu reichen unsere Ueberlegungen nicht aus, weil sie nur auf indirektem Wege zeigen, dass Grenzpunkt für z Grenzpunkt für z^* nach sich zieht und weil im Beweise von dem folgenden rein existentiellen, durch eine explizite Konstruktion nicht sicherzustellenden (und darum intuitionistisch anfechtbaren) Theorem Gebrauch gemacht wird: Wenn bei positivem Integranden das Integral $\int_0^L$ für alle L unterhalb einer festen Schranke bleibt, existiert zu beliebig vorgegebenem positiven ϵ ein l derart, dass $\int_l^L$ für alle $L > l$ kleiner als ϵ ist. Will man sich nicht auf indirekte funktionentheoretische Hilfsmittel, wie den Satz von Vitali stützen—und es ist ja mit der Zweck dieser Arbeit, solche von Nevanlinna konsequent benutzten funktionentheoretischen Ueber-

legungen durch die explizite Lösung von Integralgleichungen zu ersetzen—, so muss man einen Weg beschreiten, der die *Randwertaufgabe* heranzieht und darauf beruht, dass ihre *Green'sche Funktion ein beschränkter Kern im Sinne der Integralgleichungen ist.*

Wiederum sei $\omega = (f_\omega, g_\omega)$ die Lösung $\vartheta + w \cdot \eta$. Wenn sie an dem Ende $s = l$ des endlichen Intervalls $0 \leqq s \leqq l$ der reellen Randbedingung (20) genügt und ω^* die gleiche Bedeutung hat für den Parameterwert z^*, so bilde man $(\omega^*\omega)_s^l$ und $(\omega^*\eta)_0^s$ mittels (18); durch Auflösung nach ω^* erhält man, analog zu (27):

$$(33)\quad \begin{cases} f_\omega^*(s) = f_\omega(s) - (z - z^*)\left[f_\omega(s) \displaystyle\int_0^s \frac{D(t;\eta\omega^*)\,dt}{(z;t)\,(z^*;t)} + f_\eta(s) \int_s^l \frac{D(t;\omega\omega^*)\,dt}{(z;t)\,(z^*;t)} \right], \\ g_\omega^*(s) \text{ entsprechend.} \end{cases}$$

Hier tritt als *symmetrischer Kern* die Green'sche Funktion

$$\begin{Vmatrix} G_{pp}(s,t), & G_{pq}(s,t) \\ G_{qp}(s,t), & G_{qq}(s,t) \end{Vmatrix}$$

auf, die durch die Gleichungen

$$G_{pp}(s,t) = \begin{cases} f_\omega(s)\,f_\eta(t) & (t \leqq s) \\ f_\eta(s)\,f_\omega(t) & (s \leqq t) \end{cases}, \qquad G_{pq}(s,t) = \begin{cases} f_\omega(s)\,g_\eta(t) & (t \leqq s) \\ f_\eta(s)\,g_\omega(t) & (s \leqq t) \end{cases},$$

$$G_{qp}(s,t) = \begin{cases} g_\omega(s)\,f_\eta(t) & (t \leqq s) \\ g_\eta(s)\,f_\omega(t) & (s \leqq t) \end{cases}, \qquad G_{qq}(s,t) = \begin{cases} g_\omega(s)\,g_\eta(t) & (t \leqq s) \\ g_\eta(s)\,g_\omega(t) & (s \leqq t) \end{cases}$$

erklärt ist. Soll die Abhängigkeit von dem Parameter w der gewählten Lösung ω in Evidenz gesetzt werden, so schreiben wir $G^{(w)}$ anstelle von G. Die Gleichungen (33) lauten

$$(34)\quad \begin{cases} f_\omega^*(s) = f_\omega(s) - (z - z^*) \cdot \displaystyle\int_0^l \frac{G_{pp}(s,t)\,p(t) f_\omega^*(t) + G_{pq}(s,t)\,q(t)\,g_\omega^*(t)}{(z;t)\,(z^*;t)}\,dt, \\ g_\omega^*(s) = \cdots. \end{cases}$$

Mit zwei willkürlichen Funktionenpaaren $\xi = (x_p(s), x_q(s))$, $\eta = (y_p(s), y_q(s))$ bildet man die zu $G = G^{(w)}$ gehörige symmetrische Bilinearform $K = K^{(w)}$, bezw. ihren l-Abschnitt:

$$(35)\qquad K_l[\xi,\eta] = \int_0^l \int_0^l \frac{\sum_{\mu\,\nu} G_{\mu\nu}(s,t)\,\mu(s)\,\nu(t)\,x_\mu(s)\,y_\nu(t)}{(z;s)^2\,(z;t)^2}\,dt\,ds \qquad \begin{pmatrix} \mu = p, q \\ \nu = p, q \end{pmatrix}$$

(werden die Integrale bis ∞ ausgedehnt, so lassen wir den unteren Index l weg). Es gilt der folgende

Satz K: Die Form $K^{(w)}$ hat, wenn w ein Punkt im Grenzkreis ist, die Schranke $1/\Im z$; *d. h. es ist*

$$|K_l^{(w)}[\xi,\eta]| \leqq 1/\Im z$$

für alle l und für alle Funktionenpaare ξ und η, die den Bedingungen

$$\|\xi\|^2 = \int_0^\infty \frac{p(s)\,|\,x_p(s)\,|^2 + q(s)\,|\,x_q(s)\,|^2}{|\,z;s\,|^2}\,ds \leqq 1\,, \qquad \|\eta\|^2 \leqq 1$$

genügen.

Im Grenzpunktfall findet das Anwendung auf den Grenzpunkt w und das zugehörige $\omega = \vartheta + w\cdot\eta$. Wir bestimmen eine quadratisch integrierbare Lösung $\omega^* = \vartheta^* + w^*\cdot\eta^*$ der Differentialgleichungen (13) mit dem Parameterwert z^* dadurch, dass wir die Integralgleichungen (34) lösen, in denen G die zum Grenzpunkt w gehörige Green'sche Funktion $G^{(w)}$ und die obere Integralgrenze l durch ∞ zu ersetzen ist. *Eine solche Lösung erhält man einfach mittels der Neumann'schen Reihe.* Sie konvergiert, wenn die Schranke des Kernes < 1 ist. Dies ist der Fall, wenn

$$|\,z - z^*\,|\cdot m < \Im z \tag{36}$$

ist. Denn der Kern $\tilde{K}$ der Integralgleichungen (34) unterscheidet sich von der Form K, (35), darin, dass im Nenner je ein Faktor $(z;s)$, $(z;t)$ durch $(z^*;s)$, bezw. $(z^*;t)$ ersetzt ist. Diese Faktoren können aber in die willkürlichen Funktionen ξ und η aufgenommen werden: mit

$$\tilde{\xi} = \sqrt{\frac{(z;s)}{(z^*;s)}}\cdot\xi\,, \qquad \tilde{\eta} = \sqrt{\frac{(z;s)}{(z^*;s)}}\cdot\eta$$

ist

$$\tilde{K}_l[\xi,\eta] = K_l[\tilde{\xi},\tilde{\eta}]\,,$$

und darum folgt aus

$$|\,K_l[\xi,\eta]\,| \leqq \frac{1}{\Im z}\cdot\|\xi\|\cdot\|\eta\|$$

die Ungleichung

$$|\,\tilde{K}_l[\xi,\eta]\,| \leqq \frac{1}{\Im z}\cdot\|\tilde{\xi}\|\,\|\tilde{\eta}\| \leqq \frac{m}{\Im z}\cdot\|\xi\|\,\|\eta\|\,.$$

Die Forderung (36) lautet, wenn man den expliziten Ausdruck (25) von m benutzt:

$$\frac{2\,|\,z^* - z\,|}{|\,z^* - \bar{z}\,| - |\,z^* - z\,|} < 1\,.$$

Wählt man z fest $= z_0$ und bezeichnet dann das variable z^* wiederum mit z, so bedeutet diese Ungleichung oder

$$\left|\frac{z - z_0}{z - \bar{z}_0}\right| < r = \frac{1}{3}$$

einen nichteuklidischen Kreis um den Punkt z_0 vom Radius

$$\frac{1}{2}\lg\frac{1+r}{1-r} = \frac{1}{2}\lg 2\,.$$

Die obere Halbebene ist dabei in Poincaré'scher Weise als nichteuklidische Ebene gedeutet. *In einem nichteuklidischen Kreis um z_0, dessen Radius kleiner*

ist als die feste Grenze $\frac{1}{2}$ lg 2, *konvergiert darum die Neumann'sche Reihe gleichmässig in* z, und so ergibt sich denn ω und damit $w = f_\omega(0)$ in seiner Abhängigkeit von z als eine *analytische Funktion*. Da die obere Grenze $\frac{1}{2}$ lg 2 unabhängig ist von dem Zentrum z_0, kann man mittels derartiger Kreise von einem festen Radius $< \frac{1}{2}$ lg 2 die analytische Fortsetzung über die ganze obere Halbebene vollziehen. Man kann darum auch auf dem hier angegebenen Wege, d.i. *mittels der Randwertaufgabe* zeigen, dass der Grenzkreisfall für *ein* z_0 den Grenzkreisfall für *alle* andern z nach sich zieht.

Um Satz K zu beweisen, zeigt man zunächst, dass *bei gegebenem* l *die Ungleichung*

$$|K_l^{(w)}[\xi, \eta]| \leqq 1/\Im z \text{ für } \|\xi\|_0^l \leqq 1, \quad \|\eta\|_0^l \leqq 1 \tag{37}$$

gilt, wenn w *auf der Peripherie von* $\mathfrak{k}_l$ *liegt* (*Hilfssatz* K_l). Legt man durch einen Punkt w der Kreisscheibe $\mathfrak{k}_l$ eine Sehne, so erscheint w als der Schwerpunkt zweier Punkte w', w'' auf der Peripherie:

$$w = \tau' w' + \tau'' w''; \qquad \tau' \geqq 0, \quad \tau'' \geqq 0, \quad \tau' + \tau'' = 1.$$

Es ist dann

$$K_l^{(w)} = \tau' K_l^{(w')} + \tau'' K_l^{(w'')},$$

und so überträgt sich die Ungleichung (37) von den Punkten w der Peripherie auf die Punkte im Innern von $\mathfrak{k}_l$. Ein Punkt w des Grenzkreises liegt in *allen* Kreisen $\mathfrak{k}_l$; darum gilt für ein solches w die Ungleichung (37) identisch in l.

Um das in meiner Habilitationsschrift aufgestellte Muster zum Beweise des Hilfssatzes K_l von der dort betrachteten selbstadjungierten Differentialgleichung 2. Ordnung auf das System (13) zu übertragen, muss man es zuvor ein wenig umwandeln und vereinfachen. Ich schildere es in der modifizierten Form sogleich an dem uns hier interessierenden System (13).

Man stützt sich auf das allgemeine Kriterium: Für die aus einem stetigen komplexwertigen Kern K entspringende Bilinearform zweier willkürlicher Funktionen $x(s)$, $y(s)$:

$$K[x, y] = \int_0^1 \int_0^1 K(s, t)\, x(s)\, y(t)\, dt\, ds \qquad \left(\|x\|^2 = \int_0^1 |x(s)|^2\, ds\right)$$

gilt die Schranke

$$|K[x, y]| \leqq 1/\lambda_0 \text{ im Bereich } \|x\| \leqq 1, \quad \|y\| \leqq 1,$$

wenn das Intervall $|\lambda| < \lambda_0$ frei von Eigenwerten ist. Die Eigenwerte λ und zugehörigen Eigenfunktionen $(x(s), y(s)) \neq (0, 0)$ sind dabei erklärt durch das Gleichungspaar:

$$\begin{cases} x(s) - \lambda \int_0^1 K(s, t)\, y(t)\, dt = 0, \\ y(s) - \bar{\lambda} \int_0^1 K^\dagger(s, t)\, x(t)\, dt = 0, \end{cases} \tag{38}$$

in welchem $K^\dagger(s, t)$ der Hermitisch konjugierte Kern $\bar{K}(t, s)$ ist.[5] *Ist K symmetrisch*—nicht im Hermiteschen, sondern im Sinne der Gleichung $K(t, s) = K(s, t)$,—*so genügt es zu zeigen, dass aus*

$$x(s) - \lambda \int_0^1 K(s, t)\, \bar{x}(t)\, dt = 0 \tag{39}$$

das Verschwinden von $x(s)$ folgt, solange $|\lambda| < \lambda_0$ ist. Denn in diesem Fall lautet die zweite der Gleichungen (38):

$$y(s) - \bar{\lambda} \int_0^1 \bar{K}(s, t)\, x(t)\, dt = 0\,,$$

während sich aus der ersten ergibt

$$\bar{x}(s) - \bar{\lambda} \int_0^1 \bar{K}(s, t)\, \bar{y}(t)\, dt = 0\,.$$

Mit $\xi = (x(s), y(x))$ ist darum auch $\xi^\dagger = (\bar{y}(s), \bar{x}(s))$ eine Eigenlösung zum gleichen Eigenwert λ. Jede Eigenlösung ist die Summe einer geraden und einer ungeraden: $\xi^\dagger = +\xi$, bezw. $\xi^\dagger = -\xi$. Die geraden Eigenlösungen genügen der Gleichung (39), die ungeraden derselben Gleichung mit dem Parameter $-\lambda$ statt $+\lambda$.

Beweis des Hilfssatzes K_l: w liege auf der Peripherie von $\mathfrak{k}_l$ und (20) sei die zugehörige reelle Randbedingung, G das zugehörige $G^{(w)}$. Die erste der Integralgleichungen (38) lautet hier, wenn (x_p, x_q) für x und (y_p, y_q) für y geschrieben wird:

$$x_\mu(s) - \lambda \int_0^l \frac{\sum_\nu G_{\mu\nu}(s, t)\, \nu(t)\, y_\nu(t)}{(z; t)^2}\, dt = 0 \qquad (\mu, \nu = p, q)\,. \tag{40}$$

Der Uebergang von dem Funktionenpaar

$$\frac{\sqrt{\mu(s)} \cdot y_\mu(s)}{(z; s)} \quad \text{zu} \quad \frac{\sqrt{\mu(s)} \cdot x_\mu(s)}{(z; s)} \tag{41}$$

wird also durch den symmetrischen Kern

$$\left\| \frac{G_{\mu\nu}(s, t)\, \sqrt{\mu(s)\, \nu(t)}}{(z; s)\, (z; t)} \right\|_{\mu, \nu = p, q}$$

vollzogen. Darum darf man bei Ermittlung der Eigenwerte λ annehmen, dass in (40) die Paare (41) konjugiert-komplex sind. Dies liefert

$$x_\mu(s) - \lambda \int_0^l \frac{\sum_\nu G_{\mu\nu}(s, t)\, \nu(t)\, \bar{x}_\nu(t)}{|z; t|^2}\, dt = 0\,. \tag{42}$$

[5] Dies ergibt sich bekanntlich durch Anwendung der Theorie der Eigenwerte und Eigenfunktionen auf den Hermiteschen Kern $KK^\dagger$:

$$KK^\dagger(s, t) = \int_0^1 K(s, r)\, K^\dagger(r, t)\, dr\,.$$

Diese beiden Integralgleichungen sind äquivalent den beiden Differentialgleichungen im Intervall $0 \leqq s \leqq l$:

$$\frac{dx_p}{ds} - \frac{za''(s) - b''(s)}{za'(s) - b'(s)}\, x_q(s) = \frac{-\lambda\, q(s)\, \bar{x}_q(s)}{|\, za'(s) - b'(s)\,|^2},$$

$$\frac{dx_q}{ds} - \frac{za(s) - b(s)}{za'(s) - b'(s)}\, x_p(s) = \frac{+\lambda\, p(s)\, \bar{x}_p(s)}{|\, za'(s) - b'(s)\,|^2},$$

zusammen mit den *reellen* Randbedingungen

$$x_q(0) = 0\,, \qquad x_p(l) - h \cdot x_q(l) = 0\,.$$

Fügt man die konjugiert-komplexen Gleichungen hinzu, multipliziert die vier Gleichungen der Reihe nach mit

$$\bar{x}_q(s),\ -\bar{x}_p(s),\ -x_q(s),\ x_p(s)$$

und addiert, so ergibt sich

$$\frac{d}{ds}\left(\frac{x_p\bar{x}_q - x_q\bar{x}_p}{2i}\right) + \frac{p(s)\, D(x_p) + q(s)\, D(x_q)}{|\, za'(s) - b'(s)\,|^2} = 0\,,$$

wo

$$D(x) = \frac{1}{2i}\,\{\lambda\bar{x}^2 - \bar{\lambda}x^2 + (z - \bar{z})\, x\bar{x}\} = \Im(\lambda\bar{x}^2) + |\, x\,|^2 \cdot \Im z\,.$$

Integration nach s von 0 bis l liefert, da $(\bar{x}_p, \bar{x}_q)$ *denselben* Randbedingungen genügt wie (x_p, x_q):

$$\int_0^l \frac{p(s)\, D(x_p) + q(s)\, D(x_q)}{|\, z; s\,|^2}\, ds = 0\,. \tag{43}$$

$D(x)$ ist aber positiv-definit, wenn $|\,\lambda\,| < \Im z$. Denn führt man durch $\bar{x}^2 = |\, x\,|^2 \cdot u$ die Richtungszahl u von $\bar{x}^2$ ein, so gilt

$$D(x) = |\, x\,|^2 \cdot \Im(z + \lambda u) \geqq |\, x\,|^2 \cdot (\Im z - |\,\lambda\,|)\,,$$

weil $z + \lambda u$ den Kreis vom Radius $|\,\lambda\,|$ um den Punkt z beschreibt, während u den Einheitskreis durchläuft. Unter dieser Voraussetzung $|\,\lambda\,| < \Im z$ folgt demnach aus (43): $x_p(s) = x_q(s) = 0$, *das Intervall* $|\,\lambda\,| < \Im z$ *ist von Eigenwerten frei.*[6]

§4. Analoge Behandlung des Differenzenproblems

Die Uebertragung der in §§2–3 geschilderten Methode auf das Interpolationsproblem, von welchem wir in §1 unsern Ausgang nahmen, führt einige Komplika-

[6] Man kann auf gleiche Weise auch direkt für die Punkte w *im Innern von* $\mathfrak{k}_l$ argumentieren. Dann tritt zu (43) der positive Beitrag hinzu:

$$\left(\frac{x_p\bar{x}_q - x_q\bar{x}_p}{2i}\right)_{s=l} = |\, x_q(l)\,|^2 \cdot \Im h\,.$$

tionen mit sich, um derentwillen es nötig ist, die einzelnen Schritte zu rekapitulieren. Wir haben es zu tun mit den Differenzengleichungen (10), in denen die sechs reellen Funktionen a_n und b_n von n zunächst beliebig sind.

1. *Green'sche Formel.* Mit einem noch zu bestimmenden Faktor $H(n)$ bilden wir aus zwei Funktionenpaaren $\omega = (f, g)$, $\omega^* = (f^*, g^*)$ die Differenz

$$[H(\nu)(f(\nu)g^*(\nu) - g(\nu)f^*(\nu))]_{\nu=n-1}^{n}, \tag{44}$$

d.i. $H(n)$ mal

$$[fg^* - gf^*]_{n-1}^{n} + (1 - H(n-1)/H(n))\cdot(fg^* - gf^*)_{n-1}. \tag{45}$$

Dabei soll (f, g) den Differenzengleichungen (10) genügen, (f^*, g^*) den entsprechenden Differenzengleichungen mit dem Parameterwert z^*. Drückt man mit Hilfe dieser Gleichungen $f(n)$, $g(n)$; $f^*(n)$, $g^*(n)$ durch die Werte derselben Funktionen an der Stelle $n-1$ aus und bestimmt darauf $1 - H(n-1)/H(n)$ so, dass (45) symmetrisch in ω und ω^* wird, so kommt für (44):

$$-\frac{(z-z^*)H(n;zz^*)}{(za_n' - b_n')(z^*a_n' - b_n')}\cdot D(n;\omega\omega^*)$$

mit

$$\begin{aligned} D(n;\omega\omega^*) &= (ba' - ab')_n\cdot(ff^*)_{n-1} \\ &\quad + \tfrac{1}{2}(ba'' - ab'')_n\cdot(fg^* + gf^*)_{n-1} + (a''b' - b''a')_n\cdot(gg^*)_{n-1} \end{aligned} \tag{46}$$

ferner

$$\frac{H(n-1;zz^*)}{H(n;zz^*)} = \frac{L(n;zz^*)}{(za_n' - b_n')(z^*a_n' - b_n')}, \tag{47}$$

$$L(n;zz^*) = zz^*(a'^2 - aa'')_n - \frac{z+z^*}{2}(2a'b' - ab'' - ba'')_n + (b'^2 - bb'')_n.$$

Daraus ergibt sich vermöge der Normierung $H(0) = 1$:

$$\frac{1}{H(n;zz^*)} = \prod_{\nu=1}^{n}\frac{L(\nu;zz^*)}{(za_\nu' - b_\nu')(z^*a_\nu' - b_\nu')}. \tag{48}$$

Setzen wir wiederum

$$fg^* - gf^* = (\omega\omega^*),$$

so lautet die Green'sche Formel demnach

$$(z-z^*)\cdot\sum_{\nu=1}^{n}\frac{H(\nu;zz^*)\cdot D(\nu;\omega\omega^*)}{(za_\nu' - b_\nu')(z^*a_\nu' - b_\nu')} = H(0;zz^*)(\omega\omega^*)_0 - H(n;zz^*)(\omega\omega^*)_n. \tag{49}$$

Da $H(n;zz^*)$ unter Umständen unendlich wird, gibt man dieser Gleichung besser diejenige Gestalt, die aus ihr mittels Division durch $H(n;zz^*)$ hervorgeht: in

$$H(\nu;zz^*)/H(n;zz^*) \qquad (0 \leqq \nu \leqq n)$$

treten höchstens im Zähler, nicht im Nenner verschwindende Faktoren L auf.

Für zwei Lösungen ω, ω^* der Gleichungen (10), $z^* = z$, gilt insbesondere

$$(\omega\omega^*)_n = (\omega\omega^*)_0/H(n; zz).$$

Um ähnliche Schlüsse zu ziehen, wie im Falle der Differentialgleichungen unter der Voraussetzung (12), müssen für $\omega^* = \tilde{\omega}$ ($z^* = \bar{z}$) die Grössen $D(n; \omega\tilde{\omega})$, $H(n; z\bar{z})$ positiv werden; d.h. die quadratischen Formen der reellen Variablen x, y:

$$(ba' - ab')\,x^2 + (ba'' - ab'')\,xy + (a''b' - b''a')\,y^2 \tag{50}$$

und

$$\begin{aligned}(a'^2 - aa'')\,x^2 - (2a'b' - ab'' - ba'')\,xy + (b'^2 - bb'')\,y^2 \\ = (a'x - b'y)^2 - (ax - by)\,(a''x - b''y)\end{aligned} \tag{51}$$

müssen positiv definit sein. Diese beiden Voraussetzungen sind nicht unabhängig voneinander: die zweite ist eine Folge der ersten. Beweis: Man rechne aus, dass die zweite Form die gleiche Diskriminante hat wie die erste. Beide sind somit gleichzeitig definit. Ausserdem ist der Wert der zweiten Form für $x = b', y = a'$ gleich dem Produkt der beiden Hauptkoeffizienten

$$ba' - ab', \qquad a''b' - b''a'$$

der ersten Form, mithin positiv, wenn diese definit ist. *Wir machen also fortan betreffs der sechs Funktionen a_n, b_n die Voraussetzung, dass die Form* (50) *positiv definit ist.*

Die Unumgänglichkeit der Voraussetzung des positiven Charakters der beiden Formen (50), (51) erhellt auch daraus, dass sie nötig ist, damit die Substitution

$$w = \frac{-(za' - b')\,w' + (za'' - b'')}{(za - b)\,w' - (za' - b')}, \tag{52}$$

von der wir in §1 ausgingen, positiv imaginären Werten z und w' ein positiv imaginäres w zuordnet. Denn berechnet man $\Im w = (w - \overline{w})/2i$, so ergibt sich ein Bruch, in dessen Nenner das Quadrat des absoluten Betrages des Nenners in (52) steht, dessen Zähler aber lautet:

$$\begin{aligned}&\Im z\cdot[w'\overline{w}'(ba' - ab') - \tfrac{1}{2}(w' + \overline{w}')(ba'' - ab'') + (a''b' - b''a')] \\ &\quad + \Im w'\cdot[z\bar{z}(a'^2 - aa'') - \tfrac{1}{2}(z + \bar{z})\,(2a'b' - ab'' - ba'') + (b'^2 - bb'')].\end{aligned}$$

Ferner überzeugt man sich, dass für (11) die Form (50) in der Tat positiv definit ist. Denn ausser den schon berechneten

$$ba' - ab' = \Im\alpha\cdot\Im\gamma, \qquad a''b' - b''a' = \Im\alpha\cdot\Im\gamma\cdot|\gamma|^2$$

hat man

$$ba'' - ab'' = 2\Im\alpha\cdot\Im\gamma\cdot\Re\gamma,$$

also wird die Diskriminante von (50) gleich $4(\Im\alpha)^2\,(\Im\gamma)^4$.

Die quadratische Form $L(n; zz)$ hat wegen ihres definiten Charakters zwei konjugiert komplexe Wurzeln $\alpha_n, \bar{\alpha}_n$; α_n sei diejenige mit positivem Imaginärteil.[7] Darum gilt

$$L(n; zz) = (a'^2 - aa'')_n \cdot (z - \alpha_n)(z - \bar{\alpha}_n),$$

$$L(n; zz^*) = \tfrac{1}{2}(a'^2 - aa'')_n \cdot [(z - \alpha_n)(z^* - \bar{\alpha}_n) + (z - \bar{\alpha}_n)(z^* - \alpha_n)].$$

Es ist leicht, die Grenzfälle einzuschliessen, indem man das Positiv-Sein der Form (50) als $\geqq 0$, nicht als > 0 interpretiert.

Approximieren wir die Differentialgleichungen von §§2–3, indem wir die Variable s nur diskontinuierlich um den kleinen festen Betrag ϵ wachsen lassen, so erhalten wir unsere Differenzengleichungen—mit dem Unterschied jedoch, dass die Funktionen $a, a''; b, b''$ mit ϵ zu multiplizieren sind. Im Limes $\epsilon \to 0$ geht alsdann die Form (50) nach Division durch ϵ in die Form $p(s)x^2 + q(s)y^2$ über, die im Differentialgleichungs-Problem die entscheidende Rolle spielt, und alle Faktoren H werden zu 1.

2. *Kreis* $\mathfrak{k}_n$. w liegt auf $(\mathfrak{k}_n)$, wenn $\omega(n) = \vartheta(n) + w \cdot \eta(n)$ für n einer reellen Randbedingung genügt. Die Kreisscheibe $\mathfrak{k}_n$ ist, wie aus (49) für $\omega^* = \bar{\omega}$ folgt, gekennzeichnet durch die Ungleichung

$$\Im z \cdot \sum_{\nu=1}^{n} \frac{H(\nu; z\bar{z})\, D(\nu; \omega\bar{\omega})}{|\, za'_\nu - b'_\nu\,|^2} \leqq \Im w. \tag{53}$$

Daraus folgt, wie in §2, dass $\mathfrak{k}_n$ mit wachsendem n zusammenschrumpft.

Die Berechnung des Durchmessers d_n geschieht nach dem gleichen Muster wie in § 2. Indem man die Gleichungen

$$(\eta_1 \omega_1) = 0, \qquad (\eta_2 \omega_1) = 0$$

für w^0 direkt ausschreibt, erhält man:

$$(\bar{\eta}\vartheta_1) + iw^0 (\eta_1 \eta_2) = 0.$$

Zusammen mit der entsprechenden Gleichung für w_0 kommt so

$$\overline{(\eta\vartheta)} + i(w^0 - w_0)(\eta_1 \eta_2) = 0.$$

Es gilt

$$(\eta\vartheta)_n = 1/H(n; zz); \qquad H(n; z\bar{z}) \cdot (\eta_1 \eta_2)_n = z_2 \cdot \Sigma(n),$$

$$\Sigma(n) = \sum_{\nu=1}^{n} \frac{H(\nu; z\bar{z})\, D(\nu; \eta\bar{\eta})}{|\, za'_\nu - b'_\nu\,|^2}. \tag{54}$$

[7] Auf Grund dieser Bemerkung wird man sich wohl davon überzeugen, dass die allgemeine Theorie mit den sechs willkürlichen Funktionen a und b im Grunde nicht allgemeiner ist als die spezielle, auf dem Ansatz (11) beruhende. Die grössere Willkür kommt nur dadurch hinein, dass man auf jeder Stufe des in §1 geschilderten rekursiven Algorithmus die positive Funktion w_n noch einer gebrochenen linearen Transformation mit reellen Koeffizienten von positiver Determinante unterwerfen kann.

Der Durchmesser w_0w^0 steht jetzt nicht vertikal, für seine Grösse d_n ergibt sich

$$\Im z \cdot d_n = \frac{H(n; z\bar{z})}{|H(n; zz)|} \cdot \frac{1}{\Sigma(n)}.$$

Der erste Faktor rechts besteht aus Teilfaktoren der Gestalt

$$\frac{|L(\nu; zz)|}{L(\nu; z\bar{z})} = \frac{2|z-\alpha_\nu||\bar{z}-\alpha_\nu|}{|z-\alpha_\nu|^2+|\bar{z}-\alpha_\nu|^2} \leqq 1.$$

Setzt man

$$\left|\frac{z-\alpha_n}{\bar{z}-\alpha_n}\right| = r_n (< 1),$$

so haben wir

$$\Im z \cdot d_n = \prod_{\nu=1}^{n} \frac{2r_\nu}{1+r_\nu^2} \Big/ \Sigma(n) = \Pi(n) / \Sigma(n). \tag{55}$$

Der Durchmesser nimmt mit wachsendem n ab aus zwei Gründen: $\Pi(n)$ nimmt ab, weil immer neue Faktoren $\leqq 1$ hinzukommen; die Summe $\Sigma(n)$ nimmt zu, weil immer neue positive Summanden hinzutreten.

3. *Grenzpunkt und Grenzkreis.*

$$\frac{2r_n}{1+r_n^2} \text{ ist } = 1 - \frac{(1-r_n)^2}{1+r_n^2}.$$

Der Subtrahend rechts liegt zwischen $(1-r_n)^2$ und $\frac{1}{2}(1-r_n)^2$. Das unendliche Produkt

$$\prod_{\nu=1}^{\infty} \frac{2r_\nu}{1+r_\nu^2} = \Pi \tag{56}$$

konvergiert also oder divergiert (gegen 0), je nachdem $\Sigma(1-r_\nu)^2$ konvergiert oder divergiert. Daher ergeben sich folgende beiden Kriterien:

a) *Divergiert* $\sum_{n=1}^{\infty}(1-r_n)^2$, *so liegt der Grenzpunktfall vor.*

b) *Konvergiert hingegen* $\Sigma(1-r_n)^2$, *so liegt der Grenzpunkt- oder Grenzkreis-Fall vor, je nachdem die Summe*

$$\Sigma(\infty) = \sum_{n=1}^{\infty} \frac{H(n; z\bar{z})\, D(n; \eta\bar{\eta})}{|za_n' - b_n'|^2} \tag{57}$$

divergiert oder konvergiert.

Dass für alle Werte von z mit positivem Imaginärteil entweder simultan der Grenzkreis- oder der Grenzpunkt-Fall vorliegt, beruht darum auf den folgenden beiden Sätzen:

a) *Konvergenz und Divergenz der Reihe* $\Sigma(1-r_n)^2$ *ist unabhängig von dem Werte von* z.

b) *Konvergiert* $\Sigma(1 - r_n)^2$, *so ist die Konvergenz oder Divergenz der Summe* $\Sigma(\infty)$ *unabhängig von dem Werte von* z.

Wesentlich ist, dass die Behauptung b) nur aufgestellt wird unter der "*präliminären Bedingung*" der Konvergenz von $\Sigma(1 - r_n)^2$; und dass die beiden Faktoren $\Pi(n)$ und $1/\Sigma(n)$ in d_n, welche auf die Entscheidung einwirken, in dieser Reihenfolge a), b) und nicht in umgekehrter Folge zur Geltung gebracht werden.—Ist die präliminäre Bedingung erfüllt, so ist $\lim_{n\to\infty} r_n = 1$, d.h. die Punkte α_n verdichten sich nicht im Inneren der oberen Halbebene. Alsdann reduziert sich *auch im Grenzkreisfall für die speziellen Werte* $z = \alpha_n$ der Kreis $\mathfrak{k}_\infty$ auf einen Punkt—aber allein aus dem Grunde, weil bereits $\mathfrak{k}_n$ für diesen Wert ein Punkt ist oder weil der Faktor $2r_n/(1 + r_n^2)$ in dem *konvergenten* Produkt Π verschwindet. Für die richtige Interpretation des Satzes, dass die Unterscheidung von Grenzpunkt und Grenzkreis unabhängig ist von dem Werte von z, muss dieser Umstand beachtet werden.

Der Beweis der Behauptung a) ist nahezu trivial. Wir schreiben einen Augenblick z_0, z_0^* statt z und z^* und führen die lineare Substitution aus

$$\frac{z - z_0}{z - \bar{z}_0} = \zeta,$$

die die obere z-Halbebene in das Innere des Einheitskreises der ζ-Ebene verwandelt. Dadurch gehe insbesondere $z = \alpha_n$ in $\zeta = \zeta_n$ über; r_n ist $= |\zeta_n|$. Es handelt sich darum einzusehen, dass die Konvergenz von $\Sigma(1 - |\zeta_n|)^2$ nicht zerstört wird, wenn man ζ einer linearen, den Einheitskreis in sich überführenden Transformation unterwirft:

$$\zeta^* = \frac{\zeta - A}{1 - A\zeta} \qquad (0 \leqq A < 1).$$

Zum Beweise genügt die elementare Ungleichung

$$1 - |\zeta^*| \leqq \frac{1 + A}{1 - A}(1 - |\zeta|).$$

Der Beweis der Behauptung b) wird durch das Analogon der in §3 benutzten Integralgleichung erbracht. Man findet analog wie dort:

$$(58)\begin{cases} f_\eta^*(n) = \\ \left[f_\eta(n) + (z - z^*) \cdot \sum_{\nu=1}^{n} \dfrac{f_\eta(n)D(\nu;\vartheta\eta^*) - f_\vartheta(n)D(\nu;\eta\eta^*)}{(za_\nu' - b_\nu')(z^*a_\nu' - b_\nu')} \cdot H(\nu;zz^*) \right] \cdot \dfrac{H(n;zz)}{H(n;zz^*)}, \\ g_\eta^*(n) \text{ analog.} \end{cases}$$

Das weitere Vorgehen ist durchaus entsprechend, nur muss man noch Folgendes beachten.

1) Um mit Sicherheit zu vermeiden, dass in $H(n;zz)$ verschwindende Nenner

auftreten, muss man (bei fest gegebenem z), statt mit $n = 0$, mit einem hinreichend hohen Wert $n = n_0$ beginnen. In (48) läuft das Produkt jetzt, entsprechend der Normierung $H(n_0) = 1$, erst von $\nu = n_0 + 1$ ab, ebenso die Summe in (58). η und ϑ sollen die Randbedingungen

$$f_\eta = 1, \quad g_\eta = 0; \qquad f_\vartheta = 0, \quad g_\vartheta = 1$$

an der Stelle $n = n_0$ erfüllen. Man vergesse aber nicht, dass wir auch jetzt damit rechnen müssen, dass in $H(n; zz^*)$ verschwindende Nenner vorkommen. Wir haben deshalb dafür gesorgt, dass diese Grösse nur in Quotienten

$$H(\nu; zz^*)/H(n; zz^*) \qquad (n_0 \leqq \nu \leqq n)$$

auftritt.

2) Aus den Ausdrücken (58) für $f_\eta^*(n)$ und $g_\eta^*(n)$ bildet man die definite Form $D(n+1; \eta^*\eta^*)$. Auf der rechten Seite macht man bei Anwendung der Schwarz'schen Ungleichung wiederum Gebrauch von:

$$|\, D(\nu; \vartheta\eta^*) \,|^2 \leqq D(\nu; \vartheta\bar{\vartheta}) \cdot D(\nu; \eta^*\eta^*) .$$

Da ferner

$$|\, L(n; zz^*) \,|^2 \leqq L(n; z\bar{z}) \cdot L(n; z^*\bar{z}^*)$$

gilt, darf gleichzeitig

$$|\, H(\nu; zz^*)/H(n; zz^*) \,|^2 \qquad (n_0 \leqq \nu \leqq n)$$

durch das grössere

$$H(\nu; z\bar{z})H(\nu; z^*\bar{z}^*)/H(n; z\bar{z})H(n; z^*\bar{z}^*)$$

ersetzt werden (dies gilt selbstverständlich auch für den Faktor

$$H(n_0; zz^*)/H(n; zz^*),$$

den das vor dem Summenzeichen stehende Glied $f_\eta(n)$, bezw. $g_\eta(n)$ trägt).

3) Vor der Summation nach n ist $D(n+1; \eta^*\eta^*)$ zu multiplizieren mit

$$\frac{H(n+1; z^*\bar{z}^*)}{|\, z^* a'_{n+1} - b'_{n+1} \,|^2} = \frac{H(n; z^*\bar{z}^*)}{L(n+1; z^*\bar{z}^*)} .$$

Mit dem zusammen, was durch Quadrierung und die Ersetzung 2) aus dem Faktor hinter der eckigen Klammer in (58) geworden ist:

$$\frac{|\, H(n; zz) \,|^2}{H(n; z\bar{z}) \cdot H(n; z^*\bar{z}^*)} ,$$

gibt das

$$(59) \quad \frac{|\, H(n; zz) \,|}{H(n; z\bar{z})} \cdot \frac{|\, H(n+1; zz) \,|}{H(n+1; z\bar{z})} \cdot \frac{|\, L(n+1; zz) \,|}{L(n+1; z^*\bar{z}^*)} \times \frac{H(n+1; z\bar{z})}{|\, za'_{n+1} - b'_{n+1} \,|^2} .$$

Die ersten beiden Faktoren in (59) sind jeder

$$\leqq 1/\Pi\,, \qquad \Pi = \prod_{\nu=1+n_0}^{\infty} \frac{2r_\nu}{1+r_\nu^2}\,;$$

der dritte ist

$$\leqq \left|\frac{z-\alpha_{n+1}}{z^*-\alpha_{n+1}}\right| \cdot \left|\frac{z-\bar{\alpha}_{n+1}}{z^*-\bar{\alpha}_{n+1}}\right|.$$

Hinter dem $\times$ steht derjenige Faktor, mit welchem man $D(n+1;\,\eta\bar{\eta})$ und $D\,(n+1;\,\vartheta\bar{\vartheta})$ vor der Summation nach n multipliziert sehen möchte. Der "störende Faktor," der jetzt an die Stelle von (30) in §3 tritt, ist somit

$$\leqq \frac{1}{\Pi^2} \cdot \left|\frac{(z-\alpha_{n+1})(z-\bar{\alpha}_{n+1})}{(z^*-\alpha_{n+1})(z^*-\bar{\alpha}_{n+1})}\right|.$$

Von ihm muss noch festgestellt werden, dass er als Funktion von n beschränkt bleibt.

4) Zufolge der präliminären Bedingung und der unter 1) getroffenen Vorkehrung hat das konvergente Produkt Π einen positiven Wert, der sogar durch geeignete Wahl von n_0 so nahe wie man will an 1 herangebracht werden kann. Für

$$\left|\frac{(z-\alpha_n)(z-\bar{\alpha}_n)}{(z^*-\alpha_n)(z^*-\bar{\alpha}_n)}\right|$$

erhält man aus dem Beweise des entsprechenden Hilfssatzes in §3 die obere Schranke

$$c^2 \Big/ \left(a^2+b^2-\frac{1+r_n^2}{2r_n}\cdot 2ab\right),$$

in der a, b, c die Entfernungen z^*z, $z^*\bar{z}$, $z\bar{z}$ bezeichnen.

So ergeben sich für das Differenzenproblem die gleichen Resultate wie für das Differentialproblem in §3.

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ON COMPACT SOLUTIONS OF OPERATIONAL-DIFFERENTIAL EQUATIONS. I

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Introduction. A suggestive method of solving a partial differential equation of the type

$$\sum_{p=1}^{k} \frac{\partial}{\partial x_p}\left(\sum_{q=1}^{k} a_{pq}(x)\,\frac{\partial \varphi}{\partial x_q}\right) = \mu(x)\,\frac{\partial^2 \varphi}{\partial t^2} \qquad (\overline{a_{pq}} = a_{qp}) \tag{1}$$

in which t varies from $-\infty$ to $+\infty$, and $x = (x_1, \cdots, x_k)$ is the point of a k-dimensional domain G is to put the desired function $\varphi(t; x)$ tentatively in the form

$$\varphi(t; x) = \sum_{n} e^{i\beta_n t}\psi_n(x) \tag{2}$$

and to separate off the variable t. Denoting the operator on the left side of (1), if applied to a function $\psi(x)$ by $A\,\psi$, this leads to proper value equation

$$A\,\psi = -\beta^2 \mu(x)\psi \tag{3}$$

for the quantities $\beta = \beta_n$, $\psi = \psi_n(x)$. The solutions of (3) depend very essentially on the nature of the boundary conditions, with respect to the variable x, which were originally imposed on the solution $\varphi(t; x)$, and which, if $\varphi(t; x)$ is assumed to be of the form (2), carry over to the functions $\psi_n(x)$. From an operational standpoint the rôle of the boundary condition is to make the given operator $A\psi$ an *Hermitian* operator in the Hilbert space $\mathfrak{H}$ whose elements are the complex-valued functions $f(x)$, $g(x)$, $\cdots$ in G, for which $|f(x)|^2$, $|g(x)|^2$, $\cdots$ are Lebesgue-integrable over G, the inner product of any two elements f, g of $\mathfrak{H}$ being defined by $\int_G f(x)\,\overline{g(x)}\,dx$.[1] Consequently the most general "boundary condition" has the following form. Assuming that the given coefficients $a_{pq}(x)$ have partial derivatives of the first order (in G), consider a linear set F of $\mathfrak{H}$,

[1] For properties of Hilbert space and its operators we shall refer to the papers of J. von Neumann [11], [12], [13], [15], and to the treatise of M. H. Stone [22] listed in the bibliography at the end of the present paper.

consisting of functions $f, g, \cdots$ which have partial derivatives of the first order such that for any two elements f, g of F, the integral

$$\int_G [f \cdot \bar{A}g - Af \cdot \bar{g}]\, dx$$

$$\equiv \int_G \left[f \sum_{p,q} \frac{\partial}{\partial x_p}\left(\overline{a_{pq}} \frac{\partial \bar{g}}{\partial x_q}\right) - \bar{g} \sum_{p,q} \frac{\partial}{\partial x_p}\left(a_{pq} \frac{\partial f}{\partial x_q}\right)\right] dx \tag{4}$$

$$\equiv \int_G \left[\sum_{p,q} \frac{\partial}{\partial x_p}\left(\overline{a_{pq}} f \frac{\partial \bar{g}}{\partial x_q} - a_{pq} \bar{g} \frac{\partial f}{\partial x_q}\right)\right] dx$$

exists in any general sense and has the value zero. (If G is bounded and has a sufficiently smooth boundary, the integral (4) transforms into an integral over the boundary of G and the vanishing of (4) practically amounts to a boundary condition in the traditional way; compare [11], p. 49–50.) And $\varphi(t; x)$ satisfies the boundary condition, if for each fixed value of t, it is an element of F; in other words, if the range of the "abstract" function $\varphi_t = \varphi(t; x)$ $(-\infty < t < +\infty)$ lies entirely in F.

The method of solving (1) by separating off the variable t in the suggested way, presupposes the possibility of putting $\varphi(t; x)$ in the form (2). Now, forgetting about the variable x, the only functions $\varphi(t)$, $-\infty < t < +\infty$, which can be reasonably represented by a series of the form

$$\sum_n \psi_n e^{i\beta_n t} \tag{5}$$

are the almost periodic functions of H. Bohr (compare [6]). Bohr's definition and theory of almost periodicity can be easily adapted to functions φ_t whose values lie in $\mathfrak{H}$ (assuming the strong topology of $\mathfrak{H}$),[2] and thus we are led to the question whether the solutions of (1) have the property of almost periodicity. This question was raised and answered by Muckenhoupt [10] for the one-dimensional problem

$$\frac{\partial}{\partial x}\left(a(x) \frac{\partial \varphi}{\partial x}\right) = \mu(x) \frac{\partial^2 \varphi}{\partial t^2} \tag{6}$$

$\alpha \leqq x \leqq \beta$ in the case of the simplest boundary condition $f(\alpha) = f(\beta) = 0$. Assuming $a(x) > 0$, $\mu(x) > 0$ he could show that every solution of (6) is almost periodic (in t). Later Bochner [3] investigated the general equation (1) under general boundary conditions (somewhat different from the boundary conditions described before), also generalizing the assumption $a(x) > 0$, $\mu(x) > 0$; and he established the almost periodicity of the solutions φ_t in the general case, but only by adding a novel assumption which did not appear at all in Muckenhoupt's case. The assumption being that the range of φ_t be a *compact* set in $\mathfrak{H}$ (in the

[2] See Bochner [2]; in the following, in the actual enunciations and proofs of our theorems, we shall not make use of the details of this paper.

strong topology),[3] or, as we shall say, that φ_t be a compact solution. A few remarks will elucidate the rôle of this assumption: (i) The assumption is not a restrictive one, in other words, every almost periodic function has *eo ipso* a compact range. This follows immediately from Bochner's version of Bohr's definition of almost periodicity which, in our case, requires that every infinite sequence of real numbers $\{t_\nu\}$ contains a infinite subsequence $\{t'_\nu\}$, such that the sequence $\{\varphi_{t+t'_\nu}\}$ is uniformly convergent in $-\infty < t < +\infty$, compare [1], and [2], § 2. (ii) Our assumption is really necessary. For consider the equation $\frac{\partial^2\varphi}{\partial x^2} = \frac{\partial^2\varphi}{\partial t^2}$ in the domain G: $-\infty < x < +\infty$. If $F(x)$ is any function in $-\infty < x < +\infty$ which has a continuous second derivative and which vanishes outside a finite interval $(-\alpha, \alpha)$, then $\varphi_t = F(x+t)$ is a solution which satisfies all assumptions required in [3], except the assumption of compactness; namely, if $t_\nu = 2\nu\alpha$, then

$$\|\varphi_{t_\nu} - \varphi_{t_\mu}\|^2 = 2\int_{-\infty}^{\infty} |F(x)|^2\,dx = \text{constant} > 0 \qquad (\mu \neq \nu)$$

meaning that the sequence $\{\varphi_{t_\nu}\}$ is not compact, and thus the function φ_t not almost periodic. (iii) In the special case considered by Muckenhoupt the compactness of the solutions must, of course, be a consequence of the equation and the boundary conditions under Muckenhoupt's special assumptions, and this can also be verified directly.

We now turn our attention to equations different from (1). An important problem is the equation

$$A\varphi = \frac{\partial\varphi}{\partial t}, \qquad \varphi = \varphi(t; x, y, z) \equiv \varphi_t, \tag{7}$$

with A being $= -a\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$ as in problems of heat conduction, or $= \frac{2\pi i}{h} H$, where H is the energy-operator in Schrödinger's (compare [19]) quantum mechanical equation for non-stationary states. (In the simplest case H has the form

$$-\frac{h^2}{8\pi^2 m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + V(x, y, z),$$

so that the main difference of the two cases is the factor i in A). From what is known about the nature of the solutions of (7), it readily suggests itself (and it will follow from our general theorems) that also in the case of equation (7), for general classes of operators A, every compact solution φ_t is, of necessity, almost periodic.

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[3] This is the topology which is based on the notion of distance $\|f-g\|$. Compare, for this distance-notion, [22], p. 5, Theorem 1.2, [11], p. 65, footnote 27, and for the strong topology: [12], p. 378. We shall be concerned exclusively with the *strong* topology in $\mathfrak{H}$.

Equation (1) is a differential equation of the second degree in t, (7) is of the first degree. They are both special cases of the functional differential equation

$$\sum_{\nu=0}^{n} \frac{\partial^{n-\nu}}{\partial t^{n-\nu}} A_\nu \varphi_t = 0. \tag{8}$$

Herein φ_t is a function of t, in $-\infty < t < +\infty$, whose values are the elements of a given Hilbert space $\mathfrak{H}$, and $A_0, \cdots, A_n$, are general operators in $\mathfrak{H}$. Thus if $\mathfrak{H}$ consists of functions f, g of variables $x, y, z, \cdots$, the operators $A_\nu \varphi$ may be of an arbitrarily complicated character in these variables. With respect to the variable t, (8) is an ordinary differential equation, its degree being any integer n. We shall prove that under certain restrictions on the operators A_ν, every compact solution φ_t is almost periodic. If φ_t is (continuous and) almost periodic it can be represented by a series of the form (5), in which the β_n are real numbers and ψ_n is in $\mathfrak{H}$. Inversely if a series of this type in some grouping of its terms converges uniformly in $-\infty < t < +\infty$ then its sum is an almost periodic function; and in our case we shall prove directly that every compact solution of (8) admits of such an expansion, and, therefore, is almost periodic.

We shall use two alternative restrictions on the given operators A_ν.[4]

I. The operators A_ν and their adjoints A_ν^* commute all with each other (in particular, each operator A_ν, commuting with its adjoint, is a normal operator).[5]

II. All operators A_ν are polynomials of one Hermitian operator A.[6]

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[4] Equation (1) does not fall directly under any of the following restrictions. But it can be easily transformed into an equation to which our theorems apply. We shall deal with this point in §9, Part II.

[5] The notion of commutativity, in its literal meaning (compare, e.g. [12], p. 374) applies only to bounded operators. For general operators, the notion is more elaborate, and it was introduced and discussed in the second half of [12]. Accordingly, restriction I can be thus stated that the set $(A_0, A_1, \cdots, A_n)''$, whose elements are *always* bounded, see [12], p. 404, 405, shall be an Abelian set (i.e. that any two operators of this set shall commute). And it is this notion of commutativity which is meant in restriction I.

A more analytic way of putting restriction I is to require that each operator A_ν be representable in the form $\int\int z\, dE_\nu(z)$, $E_\nu(z)$ being a two-dimensional resolution of the identity, such that all operators $E_{\nu_1}(z_1)$, $E_{\nu_2}(z_2)$ commute (in the literal sense), compare [12], p. 410–420.

It should be noted, however, when comparing these two formulations of our restriction, that the first is entirely explicit, while in the second the $E_\nu(z)$, although uniquely determined, are defined only by implicit properties.

[6] If the A_ν are polynomials of A, then $(A_0, \cdots, A_n)' \supset (A)'$, $(A_0, \cdots, A_n)'' \subset (A)''$; hence $(A_0, \cdots, A_n)''$ is Abelian if $(A)''$ is Abelian. Now if A happens to be normal, $(A)''$ is Abelian, and hence, by footnote 5, the operators $A_0, \cdots, A_n$ and their adjoints all commute with each other.

If a Hermitian operator A is normal it is hypermaximal, and if it is hypermaximal it is normal and Hermitian. (For the notion of hypermaximality, or self-adjointness, see [22], p. 50, definition 2.11, and [11], p. 72, Def. 9). Thus we have found: if the operator A of restriction II is hypermaximal, this restriction is a special case of restriction I—but we did not require A to be hypermaximal.

Besides Restriction I or II some other restrictions of minor importance will have to be incorporated in the actual theorems which we shall fully enunciate in Part II. On the other hand, our assertion that every compact solution is almost periodic will be amplified by pertaining details.

The importance and the character of the restrictions I, II will be best appreciated if we consider some examples in which compact solutions actually fail to be almost periodic.

Consider the following special case of (8):

$$\frac{\partial}{\partial t}\varphi_t + \left(x + \frac{\partial}{\partial x}\right)\varphi_t = 0. \tag{9}$$

$\left(n = 1, A_0 = 1, A_1 = x + \dfrac{\partial}{\partial x}; -\infty < x < +\infty\right)$. Here

$$\varphi_t = e^{-x^2+xt-\frac{1}{4}t^2} \tag{10}$$

is obviously a solution. On account of

$$\|\varphi_t\|^2 = \int_{-\infty}^{\infty} e^{-2x^2+2xt-t^2}\,dx = \int_{-\infty}^{\infty} e^{-2(x-\frac{1}{2}t)^2-\frac{1}{4}t^2}\,dx = \sqrt{\frac{\pi}{2}}\,e^{-\frac{1}{4}t^2} \tag{11}$$

φ_t belongs to $\mathfrak{H}$, and is compact, since it is continuous in t and tends (strongly) to 0 for $t \to \pm\infty$. But just this convergence to 0 excludes almost periodicity.[7]

In this example $A_0 = 1$ is entirely harmless but A_1 is not normal.[8] If we multiply (9) by e^{x^2}, we obtain

$$\frac{\partial}{\partial t}e^{x^2}\varphi_t + \left(e^{x^2}\frac{\partial}{\partial x} + xe^{x^2}\right)\varphi_t = 0. \tag{12}$$

$n = 1, A_0 = e^{x^2}, A_1 = e^{x^2}\dfrac{\partial}{\partial x} + xe^{x^2}$. Here A_0, A_1 are normal, but they do not commute,[9] and (10) is again a compact solution.

Thus restriction I is really needed. It is however not always simple to decide, if a set of operators satisfies it. For instance, if all A_ν are polynomials of one

---

[7] Relation (11) exhibits directly the fact that $\|\varphi_t\|^2$ is not almost periodic, and therefore, φ_t either.

[8] An exact discussion of this operator can be given along the lines of the discussions in [22], p. 428–430 and 441. Its adjoint is $x - \dfrac{\partial}{\partial x}$.

[9] A_0 and $i\,A_1$ are hypermaximal: this is obvious for A_0, and follows for $i\,A_1$ from $x\,e^{x^2} = \frac{1}{2}\dfrac{d}{dx}(e^{x^2})$ by [22], p. 425, 428, Theorems (10.6), (10.7). (Since $\displaystyle\int_{-\infty}^{+\infty}\frac{dx}{e^{x^2}}$ is finite, the first subcase of Theorem (10.6) arises; some boundary conditions for $t \to \pm\infty$ are necessary, but it is easily seen that the solution (10) fulfills them.)

normal operator A, then I means, as was pointed out in footnote 5, that A must be hypermaximal—and the question, whether a given operator is hypermaximal or not, is in many practical instances a rather difficult one to answer (Compare [22], p. 397–614, Chapter X, which is mainly dealing with such questions). In this case, however, restriction II is effective: it states that A need only be Hermitian—for which property easily applicable formal criteria are available (see [11], 49–56, in particular footnote 2).

It remains to settle in which sense the differentiation with respect to t in the equation (8) is to be taken. From the outset there is a certain ambiguity involved. If $\varphi_t = \varphi(t; x, y, \cdots)$ we might take the differentiation as it stands, namely as a partial differentiation of $A_\nu\varphi(t; x, \cdots)$ with respect to t; or we might take it as a (properly defined) differentiation of the "abstract" function φ_t; (this latter sense was adopted in [3]). But it so happens that in important equations, there occur solutions to which none of these processes of differentiation are directly applicable.[10] On the other hand, in the general theories which exist for the solution of the equation (8), it is natural to admit as solutions certain functions $\varphi(t; x, y, z)$ to which the operator A cannot be applied directly, but to which A applies if first integrated with respect to t.[11] In order to meet these possibilities we shall replace equation (8) by an n-times integrated form, in the following manner. In Part I we shall introduce, for our "abstract" functions φ_t the notion of an integral. This will enable us to consider the expressions

$$
\begin{aligned}
I_t^m \varphi_t &= \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} \cdots \int_0^{t_{m-1}} dt_m \cdot \varphi_{t_m} \\
&= \int_0^t (t-s)^{m-1} \varphi_s \, ds
\end{aligned} \tag{13}
$$

for $m = 1, 2, \cdots$. And adding the definition

$$
I_t^0 \varphi_t = \varphi_t \tag{14}
$$

we shall replace (8) by the relation

$$
\sum_{\nu=0}^{n} A_\nu (I_t^\nu \varphi_t) = \sum_{\mu=0}^{n-1} \omega_\mu t^\mu , \tag{15}
$$

where the ω_μ are perfectly arbitrary constant elements of $\mathfrak{H}$.

---

[10] A particularly striking example of this type of behaviour is given by the solution of the wave equation in Dirac's theory of light (see Dirac [8]; an exact solution was found by Weisskopf and Wigner [23]). For this solution the inner product of φ_t and φ_0 is asymptotically $1 - c\,|\,t\,|$, in the neighborhood of $t = 0$, (c being a positive constant), and thus not differentiable at $t = 0$.

[11] In the equation of heat, if $a > 0$, the initial function $\varphi_0 = \varphi(0; x, y, z)$ may not be differentiable in the variables x, y, z. And in the case of the Schrodinger equation even such variable points φ_t have to be admitted as solutions (in order to remain in agreement with the customary interpretation of the theory, cf. [16], p. 108), where for no value of t the operator A_1 applies directly to φ_t.

Part I. Preliminaries

1. Let ψ_t be a function, defined for a set of real numbers t, whose values are elements of a Hilbert space $\mathfrak{H}$. The integration theory of Lebesgue applies to such functions to a wide extent,[12] and we are going to give a brief survey of the facts we shall need.

A function ψ_t is measurable, if for any f of $\mathfrak{H}$, the numerical function (ψ_t, f) is measurable; it is sufficient to know that this holds for the elements $f = \omega_1, \omega_2, \cdots$ of a fixed complete normalized orthogonal system, because of

$$(\psi_t, f) = \sum_{p=1}^{\infty} (\psi_t, \omega_p)\overline{(f, \omega_p)}.$$

In particular, ψ_t is measurable if it is continuous (even in the weak topology), see [12], p. 378, 379. The product of ψ_t with a numerical measurable function $a(t)$, and the inner product with another measurable function χ_t

$$(\psi_t, \chi_t) = \sum_{p=1}^{\infty} (\psi_t, \omega_p)\overline{(\chi_t, \omega_p)}$$

are again measurable. In particular,

$$\| \psi_t \| = \sqrt{(\psi_t, \psi_t)}$$

is measurable.

If, moreover, $\| \psi_t \|$ is integrable, we call ψ_t itself integrable, and for any measurable t-set T the integral $\int_T \psi_t dt$ can be characterized in the following way. We introduce, for f in $\mathfrak{H}$, the linear functional

$$L(f) = \int_T (\psi_t, f)\, dt.$$

Putting $C = \int_T \| \psi_t \|\, dt$, we have, by Schwarz's inequality,

$$| L(f) | \leqq \int_T | (\psi_t, f) |\, dt \leqq \int_T \| \psi_t \| \cdot \| f \| \cdot dt = C \cdot \| f \|;$$

therefore, by a well-known theorem of F. Riesz (comp. [11], p. 94, footnote 52), there exists a unique element f^*, such that $(f^*, f) = L(f)$. Defining $\int \psi_t\, dt$ to be this f^*,—thus

$$(\int_T \psi_t\, dt, f) = \int_T (\psi_t, f)\, dt, \tag{16}$$

we obtain an integral with the customary properties of Lebesgue integral.

2. Now let us consider these notions in the most important "realizations" of the "abstract" Hilbert space.

---

[12] See [14], 572, and [4]. In the latter paper the values of the functions are elements of a space more general than $\mathfrak{H}$, and the notions of measurability, integrability and integral, are defined in another way than in the present paper; their equivalence could be easily established.

Realization 1. Let $\mathfrak{H}$ be the space $\mathfrak{H}_0$ of all sequences $(x_1, x_2, \cdots)$ of complex numbers with a finite $\sum_{p=1}^{\infty} |x_p|^2$, the inner product of $f = (x_1, x_2, \cdots)$, $g = (y_1, y_2, \cdots)$ being $(f, g) = \sum_{p=1}^{\infty} x_p \bar{y}_p$, see [22], p. 14, and [11], p. 108. Put $\psi_t = (x_1(t), x_2(t), \cdots)$. Using the complete normalized orthogonal system $\omega_1 = (1, 0, 0, \cdots)$, $\omega_2 = (0, 1, 0, \cdots)$, $\cdots$, we see: ψ_t is measurable if and only if all $(\psi_t, \omega_p) = x_p(t)$ are measurable, and putting

$$\int_T \psi_t \, dt = (y_1, y_2, \cdots),$$

we obtain from (16), substituting $f = \omega_p$,

$$y_p = \int_T x_p(t) \, dt, \qquad (p = 1, 2, \cdots). \tag{17}$$

3. *Realization* 2. Let M be a measurable subset of a Euclidean space with a measure > 0, or a more general[13] set in which a Lebesgue measure is defined, and let $\mathfrak{H}$ be the space L_2^M of all complex-valued functions $f(P)$, which are defined in M, are measurable, and have a finite $\int_M |f(P)|^2 \, dv_P$, the inner product being $(f, g) = \int_M f(P)\overline{g(P)} \, dv_P$, see [22], p. 23–28, and [11], p. 109–111.

Consider now a function ψ_t of t. We have $\psi_t = \psi_t(P) = \psi(t, P)$, but $\psi(t, P)$ is not determined uniquely by ψ_t: we may change it arbitrarily for any t on a P-set of measure 0, that is on any (t, P)-set S, for which each S_{t_0} has the P-measure 0, if we define S_{t_0} as the set of all P for which (t_0, P) belongs to S.

If $\psi(t, P)$ is measurable in (t, P)[14] then for every $f = f(P)$ of $\mathfrak{H}$, $\psi(t, P)\overline{f(P)}$ is (t, P)-measurable and thus $(\psi_t, f) = \int_M \psi(t, P)\overline{f(P)} \, dv_P$ is t-measurable.[15] Thus ψ_t is t-measurable.

Applying Fubini's theorem in the generality of footnote 14, we find, using the Schwarz-inequality,

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[13] In the applications in the present paper, M will always be one-dimensional Euclidean.

[14] Note that we are using three notions of measure and measurability: in t which is the normal Lebesgue measure for numbers; in P, which is the assumed Lebesgue measure in M; in (t, P), which is the Lebesgue measure for the product-space of the t- and P-spaces; the latter measure being, in case M is k-dimensional Euclidean with the usual measure, the usual $(k + 1)$-dimensional (Lebesgue) measure. (For the definition of Lebesgue measure in a product-space, compare, for instance [15], p. 588.)

[15] This follows from Fubini's theorem, [7], p. 632, Theorem 4, or by considering the four expressions *max* (*real part*, 0), *max* ($-$ *real part*, 0), *max* (*imaginary part*, 0), *max* ($-$ *imaginary part*, 0), separately, from p. 630, Theorem 1. However, in all these theorems loc. cit. the integrand is assumed to be summable over the entire (t, P)-spaces. But in the second theorem this assumption is unessential. For, every real, non-negative and measurable integrand is the limit of a non-decreasing sequence of non-negative summable functions and the limit theorems on measurability and Lebesgue integrals yield the desired result (compare [7], p. 382, theorem 11, and p. 422, theorem 1; the integrands being non-negative, we admit divergent integrals attributing them the improper value $+\infty$).

$$\int_M \left(\int_T |\psi(t,P)|\,dt\right)^2 dv_P = \int_M \int_T \int_T |\psi(t,P)| \cdot |\psi(t',P)| \cdot dv_P\,dt\,dt'$$

$$= \int_T \int_T \left(\int_M |\psi(t,P)| \cdot |\psi(t',P)|\,dv_P\right) \cdot dt\,dt'$$

$$\leqq \int_T \int_T \sqrt{\int_M |\psi(t,P)|^2\,dv_P} \cdot \sqrt{\int_M |\psi(t',P)|^2\,dv_P} \cdot dt'\,dt'$$

$$= \int_T \int_T \|\psi_t\| \cdot \|\psi_{t'}\| \cdot dt\,dt' = \left(\int_T \|\psi_t\|\,dt\right)^2 .$$

Thus $\int_M \left(\int_T |\psi(t,P)|\,dt\right)^2 dv_P$ is finite, and therefore $\int_T |\psi(t,P)|\,dt$ is finite for all P, except perhaps for a P-set of measure 0. Thus $\psi(t,P)$, $|\psi(t,P)|$ are (absolutely) summable over T for every P, except perhaps for a P-set of measure 0. We can define

$$\chi(P) = \int_T \psi(t,P)\,dt\,, \qquad \chi_1(P) = \int_T |\psi(t,P)|\,dt\,,$$

and have

$$\int_P |\chi(P)|^2\,dv_P \leqq \int_P |\chi_1(P)|^2\,dv_P \leqq \left(\int_T \|\psi_t\|\,dt\right)^2,$$

and so χ, χ_1 belong to $\mathfrak{H}$. If $f = f(P)$ also belongs to M we have

$$\int_M \int_T |\psi(t,P)| \cdot |f(P)| \cdot dv_P\,dt = \int_M \chi_1(P) \cdot |f(P)| \cdot dv_P = (\chi_1, |f|)\,,$$

and thus $\psi(t,P) \cdot \overline{f(P)}$ is (absolutely) summable over the product-space of M and T. Therefore [7], p. 632, Theorem 4 applies:

$$(\chi, f) = \int_M \chi(P)\,\overline{f(P)}\,dv_P = \int_M \int_T \psi(t,P)\,\overline{f(P)}\,dv_P\,dt = \int_T (\psi_t, f)\,dt\,.$$

Comparing this with (16) we obtain $\chi = \int_T \psi_t\,dt$. In other terms

$$\left(\int_T \psi_t\,dt\right)(P) = \int_T \psi(t,P)\,dt\,; \tag{18}$$

this relation,—a close analogue to (17)—means that the integration of the "abstract" function ψ_t can be carried out simply by integrating the numerical function $\psi(t,P)$ with respect to t (for all P with the possible exception of a P-set of measure zero).

Relation (18) is effective only if $\psi(t,P)$ is (t,P)-measurable, but if ψ_t is t-measurable, $\psi(t,P)$ need not be (t,P)-measurable, because it is defined only up to a (t,P)-set S for which all S_{t_0} have the P-measure 0, and such an S may be non-measurable.[16] However, if ψ_t is measurable, there is at least one choice of $\psi(t,P)$, for which it is (t,P)-measurable. In order to prove this, we map M in a one-to-one and measure-preserving way on an interval $0 \leqq x < \alpha$

[16] Sierpinski [20] has constructed a non-measurable set of the Euclidean (t,x)-plane which intersects each line $t = t_0$ in two points only!

$(0 < \alpha \leqq +\infty)$ with its normal Lebesgue measure[17] and subsequently assume M to be the interval $0 \leqq x < \alpha$, and put $P = x$. The function

$$f_y = f_y(x) = \left\{\begin{matrix} 1, \text{ for } 0 \leqq x < y \\ 0, \text{ for } y \leqq x < \alpha \end{matrix}\right|$$

belongs to $\mathfrak{H}$ for each $y > 0$, and is y-measurable. Thus, $(\omega_1, \omega_2, \cdots$ being any complete normalized orthogonal system),

$$(\varphi_t, f_y) = \sum_{p=1}^{\infty} (\varphi_t, \omega_p)\,\overline{(f_y, \omega_p)}$$

is (t, y)-measurable. But if we use any $\varphi(t, x)$ with $\varphi_t = \varphi_t(x) = \varphi(t, x)$ (which is necessarily x-measurable for each fixed t, but need not by (t, x)-measurable),

$$(\varphi_t, f_y) = \int_0^\alpha \varphi(t, x)\,\overline{f_y(x)}\,dx = \int_0^y \varphi(t, x)\,dx\,.$$

Hence

$$\Phi(t, y) = \int_0^y \varphi(t, x)\,dx$$

is (t, y)-measurable. By the well-known theorems on differentiation of Lebesgue integrals (compare [7], p. 644–546), $\varphi(t, x)$ equals

$$\varphi_1(t, x) \equiv \limsup_{N \to +\infty} N\left(\Phi\left(t, x + \frac{1}{N}\right) - \Phi(t, x)\right)$$

---

[17] If M is any one-dimensional set and its Lebesgue measure is the normal one (as will be the case in our applications) this mapping is even unnecessary. The proof we are going to give applies literally, if the auxiliary function $f_y(x)$ is defined to be

$$f_y(x) = \left\{\begin{matrix} 1 \text{ for } x < y,\ x \text{ in } M \\ 0 \text{ for } x \geqq y,\ x \text{ in } M \end{matrix}\right|$$

and all integrals $\int_0^y$ are replaced by integrals $\int_{M(y)}$, where $M(y)$ is the set of all x in M with $x < y$. This procedure also takes care of all the subsets M of an Euclidean space, if their Lebesgue measure is the normal one; it is only necessary to map the *whole* Euclidean space in a one-to-one and measure-preserving way on the one-dimensional space. This is done in a very simple way by a Peano-curve, as was shown by Steinhaus [21], pp. 24–28. Mappings for any subset of an Euclidean space where given by F. Riesz ([18] p. 497) and J. Radon ([17], pp. 1342–1349).

For a general M, finally, the possibility of a one-to-one measure preserving mapping on an interval was communicated to one of the authors M. H. Stone, and it can be proved by using certain considerations in [15]. The considerations on p. 601–605, Chapter II, §3, loc. cit. lead to this result, if they are simplified by omitting the sets Λ_n and the variable x, restricting them to the sets K_n and the variable ξ (p. 602), and replacing the square $0 \leqq x, \xi < 1$ by the interval $0 \leqq \xi < \alpha_0$ where α_0 = total measure of M. (This construction is closely related to that one of Radon, loc. cit.)

for every fixed t, except perhaps for an x-set of measure 0 depending on t. But $\varphi_1(t, x)$ is also (t, x)-measurable and we can obviously choose $\varphi_t = \varphi_1(t, x)$.

4. So far we dealt with integration of "abstract" functions. The following preparatory lemmas are of a different nature.

LEMMA 1. *Let $A_1, A_2, \cdots, A$ be bounded linear operators in $\mathfrak{H}$ for which*

$$\lim A_n f = Af \tag{19}$$

for each f (strong topology!), and

$$\| A_n f \| \leqq C \cdot \| f \| \qquad (n = 1, 2, \cdots; f \text{ in } \mathfrak{H}) . \tag{20}$$

[18]

Then the limit relation (19) *holds uniformly in any compact set Φ of $\mathfrak{H}$.*

Proof. Otherwise there would exist an $\epsilon > 0$, two sequences $n_M \to +\infty$, $m_M \to +\infty$, and a sequence f_M from Φ, such that

$$\| A_{n_M} f_M - A_{m_M} f_M \| \geqq \epsilon, \qquad M = 1, 2, \cdots. \tag{21}$$

Replacing f_M by a convergent subsequence (Φ is compact), we may assume as well that $f_1, f_2, \cdots$ has a limit f. Now, the uniform boundedness of A_M, namely assumption (20), implies

$$\lim_{M \to \infty} (A_{n_M} f_M - A_{m_M} f_M) = Af - Af = 0,$$

$$\lim_{M \to \infty} \| A_{n_M} f_M - A_{m_M} f_M \| = 0,$$

in contradiction to (21).

LEMMA 2. *Let $\mathfrak{M}_1, \mathfrak{M}_2, \cdots$ be a sequence of mutually orthogonal closed linear manifolds in $\mathfrak{H}$, φ any element of $\mathfrak{M}_1 + \mathfrak{M}_2 + \cdots$. If φ^N is the projection of φ in $\mathfrak{M}_N$, then*

$$\varphi = \sum_{N=1}^{\infty} \varphi^N, \tag{22}$$

and the convergence of this series is uniform if φ runs over a given compact set Φ.

Proof. Denote the projection operators of $\mathfrak{M}_1, \mathfrak{M}_1 + \mathfrak{M}_2, \cdots, \mathfrak{M}_1 + \mathfrak{M}_2 + \cdots$ by $E_1, E_2, \cdots, E$ respectively ([22], p. 70, and [11], p. 74). Since $\lim E_N f = Ef$, $\varphi = E\varphi$ implies $\varphi = \lim_{N \to +\infty} E_N \varphi$, and this proves (22). Remembering $\| E_N f \| \leqq \| f \|$ (the E_N are projection operators!), the uniform convergence of (22) follows now from lemma 1.

5. If $p(z) = \sum_{\lambda=0}^{m} a_\lambda z^\lambda$ is a (complex) polynomial of the variable z, if $a_m \neq 0$, and if A is an arbitrary operator in $\mathfrak{H}$, we define $p(A)$ to be $\sum_{\lambda=0}^{m} a_\lambda A^\lambda$, its defini-

[18] Assumption (20) follows from the previous ones, see [12], p. 382, in particular footnote 35. But this is of no consequence in the present context.

tion domain being the set of all points f of $\mathfrak{H}$, for which the m successive expressions Af, $A(Af)$, $\cdots$, $A(\cdots A(Af))$ are all defined. If A is linear, $p(A)$ is obviously linear. Now, of importance is the following lemma.

LEMMA 3. *If A is a closed and Hermitian operator, and $a_m \neq 0$, the operator $\sum_{\lambda=0}^{m} a_\lambda A^\lambda$ (as just defined) is also closed.*[19]

Proof. Multiplying the polynomial-operator by $\frac{1}{a_m}$ we can obviously assume $a_m = 1$. Owing to our assumption on A, we have, for $0 < \lambda < m$

$$\| A^\lambda f \|^2 = (A^\lambda f, A^\lambda f) = (A^{\lambda+1} f, A^{\lambda-1} f) \leqq \| A^{\lambda+1} f \| \cdot \| A^{\lambda-1} f \| .$$

Hence

$$\| A^\lambda f \| \leqq \frac{\| A^{\lambda+1} f \| + \| A^{\lambda-1} f \|}{2} .$$

From this it follows, as for convex functions of a continuous variable, $(0 \leqq l \leqq m)$

$$\| A^l f \| \leqq \frac{l}{m} \cdot \| A^m f \| + \frac{m-l}{l} \cdot \| f \| .^{20} \tag{23}$$

Replacing A by θA, $\theta > 0$, (23) goes over into

$$\| A^l f \| \leqq \frac{l}{m} \theta^{m-l} \| A^m f \| + \frac{m-l}{m} \theta^{-l} \| f \| .$$

Thus corresponding to any $\epsilon > 0$, there is a $C = C(\epsilon)$, for which

$$\| A^l f \| \leqq \epsilon \| A^m f \| + C \| f \| , \qquad 0 \leqq l < m .$$

---

[19] For the definition of a closed operator see [22], p. 38, definition 2.5, and [11], p. 70, definition 5.

The condition that A be Hermitian is necessary for the validity of our Lemma. Consider in "realization 1" of our Hilbert space the operator $A f$ which is defined as follows. Denoting f by $(x_1, x_2, \cdots)$ then $A f$ is defined if $\sum_{n=1}^{\infty} n^2 |x_{2n-1}|^2$ is finite, and putting $A f = (y_1, y_2, \cdots)$, let $y_{2n-1} = 0$, $y_{2n} = n\, x_{2n-1}$. It is easy to verify that $A f$ is closed. Now, A^2 has the same domain as A, and, on its domain, A^2 is zero. But A^2 is not closed, its closure being the operator zero over the entire space.

[20] If $p_\lambda \leqq \frac{p_{\lambda+1} + p_{\lambda-1}}{2}$ then $p_{\lambda+1} - p_\lambda \geqq p_\lambda - p_{\lambda-1}$. Thus $p_\lambda - p_{\lambda-1}$ is a non-decreasing function of λ. Therefore

$$\frac{1}{m-l} \sum_{\lambda=l+1}^{m} (p_\lambda - p_{\lambda-1}) \geqq \frac{1}{l} \sum_{\lambda=1}^{l} (p_\lambda - p_{\lambda-1})$$

or

$$p_l \leqq \frac{l}{m} p_m + \frac{m-l}{m} p_0 .$$

Choosing ϵ such that

$$\left(\sum_{\lambda=0}^{m-1} |a_\lambda|\right)\epsilon \leqq \tfrac{1}{2},$$

we infer the existence of a number C' for which

$$\tfrac{1}{2}\|A^m f\| - C'\|f\| \leqq \left\|\sum_{\lambda=0}^{m} a_\lambda A^\lambda f\right\| \leqq \tfrac{3}{2}\|A^m f\| + C'\|f\|,$$

$$\|A^m f\| \leqq 2\left\|\sum_{\lambda=0}^{m} a_\lambda A^\lambda f\right\| + 2C'\|f\|.$$

Thus, if $\lim_{N\to+\infty} \|f_N\| = 0$, the relations

$$\lim_{N\to+\infty} \|A^m f_N\| = 0, \qquad \lim_{N\to+\infty} \left\|\sum_{\lambda=0}^{m} a_\lambda A^\lambda f\right\| = 0$$

are equivalent, and by (23) we also have

$$\lim_{N\to+\infty} \|A^\lambda f_N\| = 0 \text{ for } \lambda = 0, 1, \cdots, m.$$

Assume now that a sequence $\{f_N\}$ has the following properties:

$$\lim_{N\to+\infty} f_N = f, \tag{24}$$

all $\sum_{\lambda=0}^{m} a_\lambda A^\lambda f_N$ are defined (i.e., all $A^m f_N$ are defined), and

$$\lim_{N\to+\infty} \sum_{\lambda=0}^{m} a_\lambda A^\lambda f_N = f^*. \tag{25}$$

By the definition of closed operators ([22], p. 38, Definition 2.5, and [11], p. 70, Definition 5) what we have to prove is that $\sum_{\lambda=0}^{m} a_\lambda A^\lambda$ is defined also for f and has the value f^*. From (24), (25) follows

$$\lim_{M,N\to+\infty} \|f_M - f_N\| = 0, \qquad \lim_{M,N\to+\infty} \left\|\sum_{\lambda=0}^{m} a_\lambda A^\lambda (f_M - f_N)\right\| = 0,$$

and therefore, by what we have shown,

$$\lim_{M,N\to+\infty} A^\lambda (f_M - f_N) = 0, \qquad 0 \leqq \lambda \leqq m.$$

Thus, each sequence $A^\lambda f_N$ has a limit $g^{(\lambda)}$, $0 \leqq \lambda \leqq m$, of course $g^{(0)} = f$, and $\sum_{\lambda=0}^{m} a_\lambda g^{(\lambda)} = f^*$. As $A^{\lambda-1} f_N$, $A(A^{\lambda-1} f_N) = A^\lambda f_N$ have the limits $g^{(\lambda-1)}$, $g^{(\lambda)}$ and A is closed, $Ag^{(\lambda-1)}$ is defined, and $= g^{(\lambda)}$. Thus all $A^\lambda f$, $0 \leqq \lambda \leqq m$, are defined and $= g^{(\lambda)}$. Therefore $\sum_{\lambda=0}^{m} a_\lambda A^\lambda f$ is defined and $= f^*$.

Part II. The Theorems

In the present part we shall enunciate and discuss our theorems. The proofs will follow in Part III and IV.

1. *Assumptions.*

a) We consider a Hilbert space $\mathfrak{H}$, and in this space closed linear operators $A_0, A_1, \cdots, A_n$ (compare [22], p. 38, 39, and [11], p. 70), a variable point φ_t, $-\infty < t < +\infty$, and n arbitrary points $\omega_0, \omega_1, \cdots, \omega_{n-1}$ (not depending on t).

b) φ_t is integrable in any finite interval (§1, Part I), and, defining the expressions $I_t^\nu \varphi_t$ by (13) and (14), A_ν is defined for all $I_t^\nu \varphi_t$, except perhaps for a t-set of measure zero ($\nu = 0, 1, \cdots, n$), and the relation

$$\sum_{\nu=0}^{n} A_\nu (I_t^\nu \varphi_t) = \sum_{\mu=0}^{n-1} \omega_\mu t^\mu \tag{26}$$

holds.

c) The set Φ of all φ_t, $-\infty < t < +\infty$, is a compact set of $\mathfrak{H}$ (in the strong topology).

d) If ψ satisfies all relations

$$A_0\psi = A_1\psi = \cdots = A_n\psi = 0 \tag{27}$$

then $\psi = 0$.

THEOREM I. *If the assumptions a) — d) hold and if the operators A_ν and their adjoints A_ν^* commute all with each other*[5] *then the following statements are true.*

There is a (finite or countably infinite) sequence of mutually orthogonal closed linear manifolds $\mathfrak{M}_1, \mathfrak{M}_2, \cdots$ with $\mathfrak{M}_1 + \mathfrak{M}_2 + \cdots = \mathfrak{H}$ such that

A. *Every A_ν is reduced by every $\mathfrak{M}_N$*[21]

B. *Every A_ν is, in every given $\mathfrak{M}_N$, a bounded operator (and therefore defined throughout $\mathfrak{M}_N$)*[21]

C. *Except perhaps for a t-set of measure 0, φ_t can be expanded into a series*

$$\varphi_t = \sum_N \varphi_t^N, \qquad \varphi_t^N \text{ belongs to } \mathfrak{M}_N \text{[22]} \tag{28}$$

which (if infinite at all) converges uniformly for all t.

D. *Each φ_t^N can be written as a finite sum, $l_N \leqq n$,*

$$\varphi_t^N = \sum_{q=1}^{l_N} e^{i\beta_{N,q}t} \psi_{N,q}, \tag{29}$$

where $\psi_{N,q}$ belongs to $\mathfrak{M}_N$ and is independent of t, and $\beta_{N,q}$ is a real number.

E. *The following equations hold.*

$$\sum_{\nu=0}^{n} (i\beta_{N,q})^{n-\nu} A_\nu \psi_{N,q} = 0, \qquad N = 1, 2, \cdots; \; q = 1, \cdots, l_N. \tag{30}$$

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[21] Compare [22], p. 150, 151, and [11], p. 78–80.

[22] Thus φ_t^N is the projection of φ_t into $\mathfrak{M}_N$.

THEOREM II. *If the assumptions a)–d) hold, and the operators A are polynomials of one Hermitian operator* A[6]

$$A_\nu = \sum_{\lambda=0}^{m} a_{\nu,\lambda} A^\lambda \qquad \text{(the } a_{\nu,\lambda} \text{ are complex numbers)}, \tag{31}$$

not all of which are constants, and if the equations

$$\sum_{\nu=0}^{n} (i\beta)^{n-\nu} a_{\nu,\lambda} = 0, \qquad \lambda = 0, \cdots, m \tag{32}$$

have no real root β *in common, then the following statements are true.*

There is a closed linear manifold $\mathfrak{M}$ *such that:* (i) A *is reduced by* $\mathfrak{M}$,[21] (ii) A *is in* $\mathfrak{M}$ *hyper-maximal*[6] *and has there a pure point-spectrum,*[23] (iii) *except perhaps for a t-set of measure* 0, φ_t *belongs to* $\mathfrak{M}$, *and* (iiii) *there is a (finite or countably infinite) sequence of mutually orthogonal closed linear manifolds* $\mathfrak{M}_1, \mathfrak{M}_2, \cdots$ *with* $\mathfrak{M}_1 + \mathfrak{M}_2 + \cdots = \mathfrak{M}$ *(not* $\mathfrak{H}$*), such that the assertions A. – E. hold in literally the same form as in theorem* I.

THEOREM III. *If the assumptions a)-c) hold, and the operators* A_ν *are polynomials of one Hermitian operator* A, *see* (31), *not all of which are constants, and if the equations* (32) *have no real root* β *in common, than the following statements are true.*

There is a closed linear manifold $\mathfrak{M}$, *such that*

A′. A *is reduced by* $\mathfrak{M}$.[21]

B′. A *is in* $\mathfrak{M}$ *hypermaximal*[6] *and has there a pure point-spectrum.*[23]

C′. *Except perhaps for a t-set of measure* 0, φ_t *belongs to* $\mathfrak{M}$.

2. Our theorems are the more true, if the underlying space is not the countably-dimensional Hilbert space, but a finite dimensional Euclidean space (with the other properties of Hilbert space). This will be clear from the details of the proofs. But the Euclidean case can also be directly reduced to the case of a proper Hilbert space by adding a countable infinity of components to the Euclidean space and extending the given operators $A_0, A_1, \cdots, A_n$ by the convention that they be zero on the space elements which are formed by the added components.

Incidentally, if the space is finite-dimensional, the enunciations and proofs of our theorems may be simplified considerably—but we are not primarily interested in this case.

3. As regards assumption (31) its precise meaning is the following. If $p(z) = \sum_{\lambda=0}^{m} a_\lambda z^\lambda$ is a (complex) polynomial of the variable z, *if* m_1 *is the greatest* λ *with* $a_\lambda \neq 0$, and if A is an arbitrary operator in $\mathfrak{H}$; then we define $p(A)$ to be $\sum_{\lambda=0}^{m} a_\lambda A^\lambda$ its definition domain being the set of all points f of $\mathfrak{H}$ for which

[23] That is, there exists a normalized orthogonal system $\tilde{\omega}_1, \tilde{\omega}_2, \cdots$, which determines the closed linear manifold $\mathfrak{M}$, and a sequence of real numbers $w_1, w_2, \cdots$ such that $A\tilde{\omega}_p = w_p\tilde{\omega}_p$ for $p = 1, 2, \cdots$.

the m_1 successive expressions Af, $A(Af)$, $\cdots$, $A(\cdots A(Af))$ are all defined. Using *this* definition of the domain of $p(A)$, we proved in Lemma 3 that for a closed Hermitian A, $p(A)$ is also closed. As lemma 3 will be needed essentially, we observe that the introduction of the exact degree m_1 of the polynomial $p(z)$ is essential for its validity.

4. Assumption *d*) in Theorem I and II cannot be dispensed with. Namely if there is a common non-trivial solution ψ of the equations (27), then, for every bounded measurable numerical function $c(t)$, the function $\psi_t = c(t)\psi$ is a compact solution of our equation, but ψ_t is not almost periodic unless $c(t)$ is almost periodic.

5. The condition, in Theorem II and III, that the equations (32) have no real root β in common, is likewise necessary. If β were a common root, each $\varphi_t = e^{i\beta t}\psi$ would satisfy the assumptions of the theorem, provided that ψ is so chosen that $A\psi$, $A^2\psi$, $\cdots$, $A^m\psi$ are all defined and A would remain entirely unrestricted. (For instance, it could be hypermaximal, with a pure continuous spectrum). On the other hand it is not difficult to formulate the changes which the abandoning of this condition would bring about in the theorems. They are not essential, and we do not propose to pursue the subject beyond our statements.

6. As regards the rôle of our theorems, we observe that the emphasis lies on Theorem I and Theorem II, not on Theorem III. Theorem III, which does not employ assumption *d*), does not assert the statements $C - E$, or any other statements implying the almost periodicity of φ_t, and therefore, has no interest of its own in the present context. We enunciated Theorem III for technical reasons only; namely, Theorem II results from a combination of Theorem I and Theorem III. In part II and III we shall give detailed proofs of the latter theorems, and we are going to show now that Theorem II is a simple consequence of them. If the hypotheses of Theorem II hold, then, by Theorem III, there exists a closed linear manifold $\mathfrak{M}$ with respect to which the assertions (i)–(iii) of Theorem II hold true. On account of this, substituting the space $\mathfrak{M}$ for $\mathfrak{H}$ in Theorem I, all conditions of this theorem are satisfied, and the statements of this theorem are precisely the assertion (iiii) of Theorem II.

Incidentally, in Theorem II, the relations (30) can be written

$$\sum_{\lambda=0}^{m}\left(\sum_{\nu=0}^{n}(i\beta_{N,q})^{n-\nu}a_{\nu,\lambda}\right)A^{\lambda}\psi_{N,q}=0\,, \qquad \begin{matrix} N=1,2,\cdots,\\ q=1,\cdots,l_N.\end{matrix}$$

Using the $\tilde{\omega}_q$ of footnote 23 we see that, expanding $\psi_{N,q}$ in an orthogonal series of the $\tilde{\omega}_1$, $\tilde{\omega}_2$, $\cdots$ only those $\tilde{\omega}_p$ can occur for which

$$\sum_{\lambda=0}^{m}\left(\sum_{\nu=0}^{n}(i\beta_{N,q})^{n-\nu}a_{\nu,\lambda}\right)w_p^{\lambda}=0\,. \tag{33}$$

By one of our hypotheses this polynomial in w_p does not vanish identically, hence there exists only a finite number of w_p, the number being $\leqq m$. Thus every $\psi_{N,q}$ is a linear aggregate of terms belonging to not more than m different proper values w_p.

7. Thus, the method of finding the compact solutions by separating off the variable t, as described at the beginning of the introduction, is not precisely legitimate for general equations of the form (8), but it comes very near the actual facts in case the operators A_ν are polynomials of one Hermitian operator.

8. Of special interest is the equation

$$\frac{\partial^n \varphi}{\partial t^n} = A\varphi \tag{34}$$

which includes equation (7), $n = 1$, and equation (1) for $\mu(x) \equiv 1$, $n = 2$, (of course, (34) is to be understood in the integrated form corresponding (26)). In this case $A_0 = 1$, $A_n = A$, and $A_\nu = 0, 1 \leqq \nu \leqq n-1$. So that, according to Theorems I and II, if A is normal, or Hermitian, every compact solution φ_t is a uniformly convergent sum of expressions (29), whose "elementary" parts $e^{i\beta t}\psi$ are themselves solutions of the given equation. In particular, each $(i\beta)^n$ is a proper value of the operator A. In case n is odd, $(i\beta)^n$ is pure imaginary, and thus can only occur if A is not hypermaximal. This is in keeping with the known fact, that the solutions of the heat equation cannot "as a rule," be composed of discrete elementary oscillations.

9. For a general $\mu(x)$, equation (1) has the form

$$\frac{\partial^2 A_0 \varphi_t}{\partial t^2} = A\varphi_t\,, \tag{35}$$

where $A_0 = \mu(x)$, and, unless $\mu(x)$ is of a very special form, our theorems, as they stand, will not apply. But usually $\mu(x) > 0$, and this implies that A_0 is positive-definite. Thus we can form $A_0^{-\frac{1}{2}}$. If we even have $0 < a \leqq \mu(x) \leqq b < +\infty$, then $A_0^{\frac{1}{2}}$, $A_0^{-\frac{1}{2}}$ are bounded, and thus we can form the operators $A'_\nu = A_0^{-\frac{1}{2}} A_\nu A_0^{-\frac{1}{2}}$ for $\nu = 0, 1, 2, \cdots, n (A'_0 = 1)$, and at the same time the transformation $A_0^{\frac{1}{2}}\varphi = \psi$ leaves the compactness of φ_t unaffected, and its inverse $\psi = A_0^{-\frac{1}{2}}\varphi$ the almost periodicity of the resulting ψ_t. Now multiplication by $A_0^{-\frac{1}{2}}$ carries (8) into

$$\sum_{\nu=0}^{n} \frac{\partial^{n-\nu}}{\partial t^{n-\nu}} A'_\nu \psi_t = 0\,.$$

For instance, in the case of equation (35), if A is Hermitian, then so is A', and Theorem II does apply.

Part III. Proof of Theorem I

1. We shall assume *throughout this part*, without further mention, that the operators $A_0, \cdots, A_n$ and the function φ_t satisfy the hypotheses of theorem I. We shall treat Theorem I by various steps, and we first dispose of a special case.

LEMMA 4. *Theorem* I *holds if* $\mathfrak{H}$ *stands for a* 1*-dimensional space* (see §2, part II).

Proof. Let ψ_0 be an element of space, $\neq 0$; then $\varphi_t = a(t)\psi_0$; and the compactness of φ_t means that the numerical function $a(t)$ is bounded. Every operator is simply a multiplication by a constant, thus $A_\nu\chi = c_\nu\chi$. Hence (26) reads

$$\sum_{\nu=0}^{n} c_\nu I_t^\nu a(t) = \sum_{\mu=0}^{n-1} b_\mu t^\mu .$$

(Here, as subsequently in similar statements, a t-set of measure zero may be excepted.) This is an inhomogeneous differential equation for $I_t^n a(t)$ with constant coefficients. Therefore $I_t^n a(t)$, and consequently also $a(t)$ itself, is a finite linear aggregate of terms of the form $t^\rho e^{(\gamma+i\beta)t}$ ($\rho = 0, 1, 2, \cdots$; β, γ real). Since $a(t)$ is bounded, all ρ and γ are zero, hence

$$a(t) = \sum_{q=1}^{l} b_q e^{i\beta_q t},$$

where

$$\sum_{\nu=0}^{n} (i\beta_q)^{n-\nu} c_\nu = 0 ;$$

and the statements $A - E$ (of theorem I) hold for $\mathfrak{M}_1 = \mathfrak{H}$ (no other $\mathfrak{M}_N$'s),

$$\varphi_t^1 = \varphi_t, l_1 = l, \psi_{1,q} = b_q\psi_0, \beta_{1,q} = \beta_q .$$

2. Let $\mathfrak{N}$ be any closed linear manifold of $\mathfrak{H}$ which reduces all A_ν. Then the hypotheses of Theorem I are also true in $\mathfrak{N}$ (that is for the part of $A_0, \cdots, A_n$ in $\mathfrak{N}$ and the projection of φ_t in $\mathfrak{N}$). This is obvious for the assumptions *a*), *b*), *d*), and also for *c*) because the projection of $\mathfrak{H}$ into $\mathfrak{N}$ is a continuous operation; and that the A_ν and their adjoints commute also in $\mathfrak{N}$ is a consequence of the condition that $\mathfrak{N}$ reduces all A_ν.[24] This enables us to formulate the following lemma.

LEMMA 5. *Let* $\mathfrak{N}_1, \mathfrak{N}_2, \cdots$ *be a (finite or infinite) sequence of mutually orthogonal closed linear manifolds, with* $\mathfrak{N}_1 + \mathfrak{N}_2 + \cdots = \mathfrak{H}$, *each* $\mathfrak{N}_N$ *reducing all* A_ν. *If the statements A-E of Theorem* I *hold for each* $\mathfrak{N}_N$, *then they also hold in the entire* $\mathfrak{H}$.

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[24] The Abelian character of the set $(A_0, \cdots, A_n)''$, see footnote 5, carries over from $\mathfrak{H}$ to any $\mathfrak{N}$ which reduces the A_ν.

Proof. Denoting, for each $\mathfrak{N}_N$, the $\mathfrak{M}_1, \mathfrak{M}_2, \cdots$ of A-E by $\mathfrak{M}^*_{M,1}, \mathfrak{M}^*_{M,2}, \cdots$, call all $\mathfrak{M}^*_{M,N}$ (in any order) $\mathfrak{M}_1, \mathfrak{M}_2, \cdots$. They satisfy A-E in the entire $\mathfrak{H}$. This is obvious for A, B, D, E; also for C if lemma 2 is taken into consideration.

3. Lemmas 4 and 5 will be instrumental in reducing the proof of the general Theorem I to the proof of highly specialized cases.

Let $E_\nu(z)$ be the resolution of identity belonging to A_ν,[25] $\nu = 0, 1, \cdots, n$; z complex. They all commute with each other. Put

$$F_N = \prod_{\nu=0}^{n} [E_\nu(N + iN) - E_\nu(N - iN) - E_\nu(-N + iN) + E_\nu(-N - iN)].$$

Then a simple direct computation, using the definitory properties of the operators $E_\nu(z)$,[25] shows that $F_N F_M = F_N$ for $N \leqq M$. In particular $F_N F_N = F_N$ and $F_N F_{N+1} = F_N$; this means that each F_N is a projection operator, and that $F_1 \leqq F_2 \leqq F_3 \leqq \cdots$. Therefore the operators $G_1 = F_1$, $G_2 = F_2 - F_1$, $G_3 = F_3 - F_2, \cdots$ are again projections; let $\mathfrak{N}_1, \mathfrak{N}_2, \cdots$ be the closed linear manifolds belonging to $G_1, G_2, \cdots$. The $G_1, G_2, \cdots$ and $E_\nu(z)$ commute all with each other, thus $\mathfrak{N}_1, \mathfrak{N}_2, \cdots$ reduce all $E_\nu(z)$, and therefore all A_ν. Obviously $F_1 \leqq F_2 \leqq \cdots$, $\lim F_M = 1$, therefore the $G_1, G_2, \cdots$ are mutually orthogonal and $G_1 + G_2 + \cdots = 1$, that is $\mathfrak{N}_1, \mathfrak{N}_2, \cdots$ are mutually orthogonal and $\mathfrak{N}_1 + \mathfrak{N}_2 + \cdots = \mathfrak{H}$.

If we write (symbolically)

$$A_\nu = \iint z dE_\nu(z)$$

we have (see [11], p. 115, 116, theorem 11*, and [12], p. 410–412, theorem 17)

$$F_M A_\nu F_M = \int_{-M \leqq \Re(z) \leqq M} \int_{-M \leqq \Im(z) \leqq M} z dE_\nu(z),$$

thus $F_M A_\nu F_M$ is bounded. But, owing to $G_M \leqq F_M$ (that is $F_M G_M = G_M F_M = G_M$),

$$G_M A_\nu G_M = G_M (F_M A_\nu F_M) G_M;$$

hence $G_M A_\nu G_M$ is bounded too. Considering that A_ν is reduced by $\mathfrak{N}_M$, the latter means that A_ν is bounded in $\mathfrak{N}_M$. And applying lemma 5 we find that, in the proof of Theorem I, we may restrict ourselves to bounded A_ν's, thus:

Specialization 1. We may assume that each A_ν is bounded.

4. Now if $A_0, \cdots, A_n$ are bounded and they and their adjoints commute with each other, there exists a bounded Hermitian operator A of which they are all functions:

$$A_\nu = F_\nu(A) \tag{36}$$

[25] See footnote 5, and [12], p. 410–412, theorem 17.

(compare [13], p. 192, 213, theorem 6). The $F_\nu(\lambda)$ are bounded complex-valued Baire functions of the real variable λ.

We choose a *maximal* normalized orthogonal system consisting of proper functions $\psi_1, \psi_2, \cdots$ of A; we denote by $[\psi_l]$ the (closed) linear manifold determined by ψ_l, and put $\mathfrak{N} = [\psi_1] + [\psi_2] + \cdots$. Each $[\psi_l]$ obviously reduces A; consequently, $\mathfrak{N}$ and $\mathfrak{H} - \mathfrak{N}$ also reduce A; and, in further consequence, $[\psi_1]$, $[\psi_2]$, $\cdots$, $\mathfrak{N}$, $\mathfrak{H} - \mathfrak{N}$ reduce also all the operators $A_\nu = F_\nu(A)$. No proper function of A can be orthogonal to all $\psi_1, \psi_2, \cdots$, that is belong to $\mathfrak{H} - \mathfrak{N}$. Thus A in $\mathfrak{H} - \mathfrak{N}$ has a pure continuous spectrum. Therefore, applying lemma 5 to the manifolds $\mathfrak{H} - \mathfrak{N}$, $[\psi_1]$, $[\psi_2]$, $\cdots$, and disposing of the manifolds $[\psi_1]$, $[\psi_2]$, $\cdots$ with the aid of lemma 4, we are led to

Specialization 2. We may assume

$$A_\nu = F_\nu(A), \tag{36}$$

where the $F_\nu(\lambda)$ are bounded complex-valued Baire functions of the real variable λ, and A is a bounded Hermitian operator with a pure continuous spectrum.

5. We now apply the Hellinger-Hahn analysis of continuous spectra, in the version of [22], Chapter VII. According to this theory, if a Hermitian operator A has a pure continuous spectrum, there exists a sequence of mutually orthogonal closed linear manifolds $\mathfrak{N}_1, \mathfrak{N}_2, \cdots$, with $\mathfrak{N}_1 + \mathfrak{N}_2 + \cdots = \mathfrak{H}$, each $\mathfrak{N}_M$ reducing A, such that, in each $\mathfrak{N}_M$, A is a *simple* operator.[26] Thus, by lemma 5,

Specialization 3. The operator A, in specialization 2, may be assumed to be simple.

6. Now, any simple Hermitian operator A in $\mathfrak{H}$ can be exhibited in the following "realization."[27] There exists a monotone non-decreasing function $\rho(\lambda)$, $-\infty < \lambda < +\infty$; $\mathfrak{H}$ is the space L_2^* of all complex-valued ρ-measurable functions $\mathfrak{f}(\lambda)$, (any two $\mathfrak{f}(\lambda)$ differing only on a λ-set of ρ-measure zero are not different elements of Hilbert space), for which $\int_{-\infty}^{+\infty} |\mathfrak{f}(\lambda)|^2 \, d\rho(\lambda)$ is finite, the inner product being

$$(\mathfrak{f}, \mathfrak{g}) = \int_{-\infty}^{+\infty} \mathfrak{f}(\lambda) \overline{\mathfrak{g}(\lambda)} \, d\rho(\lambda);$$

and in this realization of $\mathfrak{H}$, the operator A assumes the form

$$A\mathfrak{f}(\lambda) = \lambda \mathfrak{f}(\lambda). \tag{37}$$

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[26] For the definition of a simple operator see [22], p. 275, and, for the proof of our statement, p. 295, theorem 7.17. In the statement (3) of this theorem a diagonal matrix would mean that A has proper-functions, which is excluded by its having a pure point-spectrum. Thus the matrices in (3) are all "Jacobi forms," but these are simple by p. 286, 287, theorem 7.14.

[27] See [22], p. 226, theorem 6.2; p. 268, Lemma 7.1; p. 277, Theorem 7.10; our A, φ, L_2^* are called there H, f, $L_2(f)$.

The "distribution-function" $\rho(\lambda)$ stands in close relation to the resolution of identity $E(\lambda)$ pertaining to A; namely, there exists an element φ of $\mathfrak{H}$, such that $\rho(\lambda) = \| E(\lambda)\varphi \|^2$. This relation shows that $\rho(\lambda)$ is continuous, since, A having a pure continuous spectrum, $E(\lambda)$ is continuous (see [22], p. 189, 190, theorem 5.13). Moreover, in our case, A is also bounded; this implies the existence of finite numbers λ_1, λ_2, $(\lambda_1 < \lambda_2)$, such $\| E(\lambda)\varphi \| = 0$ for $\lambda < \lambda_1$, $\| E(\lambda)\varphi \| = \| \varphi \|$ for $\lambda \geqq \lambda_2$. In other words, $\rho(\lambda) = 0$ for $\lambda < \lambda_1$, $\rho(\lambda) =$ const. for $\lambda \geqq \lambda_2$.

Let $\lambda = \sigma(x)$, $0 \leqq x < \alpha_0 = \| \varphi \|^2$ denote the inverse of the function $x = \rho(\lambda)$, $\lambda_1 \leqq \lambda \leqq \lambda_2$; its values in the intervals of constancy of $\rho(\lambda)$ will not matter, as they are sets of ρ-measure 0, and so we may define $\sigma(0) = \lambda_1$, $\sigma(\alpha_0) = \lambda_2$, and $\sigma(x) = \bar{\lambda}$ in an interval of constancy $\underline{\lambda} < \lambda < \bar{\lambda}$. Putting $f(x) = \mathfrak{f}(\sigma(x))$, (37) becomes

$$Af(x) = \sigma(x)f(x), \tag{38}$$

where $\mathfrak{H}$ is realized as the space L_2^M of §3, Part I, the set M being the interval $0 \leqq x < \alpha_0$. In this realization we have simply

$$F(A)f(x) = F(\sigma(x))f(x).$$

This is obvious for polynomials $F(\lambda)$; thus it holds for all Baire functions $F(\lambda)$, in particular for our functions $F_\nu(\lambda)$ (see [13], p. 205, statements *a*)–*h*), and p. 196, theorem 3). Putting $F_\nu(\sigma(x)) = G_\nu(x)$, (36) goes over into

$$A_\nu f(x) = G_\nu(x)f(x); \tag{39}$$

and $G_\nu(x)$ is again a Baire function, since $\sigma(x)$ is monotonous. This leads to the following specialization.

Specialization 4. We may—and we shall—restrict ourselves to the following case. The space $\mathfrak{H}$ is realized as L_2^M (Realization 2, §3, Part I), the set M being a measurable set of real numbers, with a finite positive measure;[28] and, in this realization,

$$A_\nu f(x) = G_\nu(x)f(x), \tag{39}$$

where $G_0(x), \cdots, G_n(x)$ are bounded complex-valued Baire functions in M.

7. If our problem, namely the proof of Theorem I, appears in the form of Specialization 4, we may easily replace the set M by any set M^* which differs from it only in a set of measure 0. This is a consequence of the fact that functions of L_2^M, and functions $G_\nu(x)$, differing only on an x-set of measure 0, are not considered to be essentially different.

8. LEMMA 6. *Let M_1, M_2, $\cdots$ be a (finite or infinite) sequence of mutually exclusive measurable subsets of M, with $M_1 + M_2 + \cdots = M$. If Theorem* I

[28] We do not insist that M be an interval $0 \leqq x < \alpha_0$ as it automatically turned out to be in the argument leading to (39).

holds in each M_N (that is, considering the relation (39) *and the function $\varphi_t = \varphi(t, x)$ on the x-set M_N only), it holds also in M.*

Proof. If P is a subset of M of positive measure, the functions $f(x)$ of L_2^M which vanish in $M - P$ form a closed linear manifold in L_2^M which may be identified with L_2^P. Thus, putting $\mathfrak{M}_N \mid= L_2^{M_N}$, our lemma follows from lemma 4.

9. If Specialization 4 prevails, the abstract function φ_t may be realized by a function $\varphi(t, x)$ which is (t, x)-measurable (§3, Part I), and

$$I_t^m \varphi_t = J_t^m \varphi(t, x)$$

where

$$J_t^0 \varphi(t, x) = \varphi(t, x)$$

$$J_t^m \varphi(t, x) = \int_0^t \varphi(s, x)(t - s)^{m-1}\, ds\,, \qquad m = 1, 2, \cdots,$$

and (26) reads:

$$\sum_{\nu=0}^{n} G_\nu(x)\, J_t^\nu \varphi(t, x) = \sum_{\mu=0}^{n-1} \omega_\mu(x)\, t^\mu\,, \tag{40}$$

valid for any (t, x), except a t-set T of measure 0, and, for each t not in T, an x-set of measure 0 (depending on t). By Fubini's theorem ([7], p. 627, theorem 1), the total excepted (t, x)-set is of measure 0, and, consequently, it is composed of an x-set X of measure 0, and, corresponding to each x not in X, of a t-set of measure 0 (depending on x). By §7, the set X may be disregarded (that is, it may be assumed to be void). The set $\bar{N}$ of those points x for which all $G_\nu(x)$ vanish for $\nu = 0, \cdots, n$ has also the measure 0. Otherwise the function

$$\varphi(x) = \begin{cases} 1 \text{ for } x \text{ in } \bar{N} \\ 0 \text{ for } x \text{ in } M - \bar{N} \end{cases}$$

would violate the assumption *d*) underlying Theorem I. Omitting the set $\bar{N}$ too, relation (40) is for each fixed x an ordinary differential equation in t (of a degree k lying between 0 and n), with constant coefficients, for the function $J_t^n \varphi(t, x)$, whose right side is a polynomial in t (valid with the possible exception of a t-set of measure 0, depending on x).

Applying to (40) one more integration $\int_0^t$, we obtain an equation which is valid without any t-exceptions. This integration does not at all alter the form of equation (40); it only raises the order from n to $n + 1$, replacing J_t^ν by $J_t^{\nu+1}$ $(\nu = 0, 1, \cdots, n)$, $G_\nu(x)$ by $G_{\nu+1}(x)$, $(\nu = 0, 1, \cdots, n)$, and makes the new $G_0(x)$ zero. Thus, in the new equation, since no $J_t^0 \varphi\,(t, x)$-term is present, all occurring terms are continuous with respect to t. Therefore, this differential equation (with respect to t) can be solved in the customary manner, and $J_t^{n+1} \varphi(t, x)$ is a linear aggregate of a finite number of terms of the form

$$t^\rho e^{(\gamma + i\beta)t} \qquad \rho = 0, 1, \cdots; \gamma, \beta \text{ real.} \tag{41}$$

Differentiating this expression for $J_t^{n+1}\varphi(t, x)$ $(n+1)$-times with respect to t, we find that $\varphi(t, x)$ is also a finite linear aggregate of terms (41); but, this time, a t-set of measure zero, depending on x, has to be excepted. The quantities ρ,γ,β will depend on x, each number $z = \gamma(x) + i\beta(x)$ being a root of

$$\sum_{\nu=0}^{n} z^{n-\nu} G_\nu(x) = 0. \tag{42}$$

$\rho = \rho(x)$ is less than the multiplicity of this root z, therefore $\rho = 0, \cdots, n-1$.

Consider the number 0 and the different real parts of all roots of (42). (In what follows multiple roots of (42) will always be counted as one root.) Arrange these numbers in an increasing sequence giving the number 0 the index zero. Thus the indices are some numbers between $-n$ and n. Complete this sequence to one in which the indices run from $-n$ to n, by inserting before its first term additional terms differing from it by 1, 2, $\cdots$, and after its last term additional terms differing from it by 1, 2, $\cdots$, in such a way that a monotone sequence

$$\gamma_{-n}(x) < \gamma_{-n+1}(x) < \cdots < \gamma_{-1}(x) < \gamma_0(x) = 0 < \gamma_1(x) < \cdots < \gamma_n(x)$$

results. In a similar way form a sequence

$$\beta_1(x) < \beta_2(x) < \cdots < \beta_n(x),$$

containing the different imaginary parts of the roots, $\beta_1(x)$ being the smallest imaginary part, and $\beta_n(x)$ being the greatest, or differing from the greatest by 1, or 2, or $\cdots$. All these functions $\gamma_h(x)$, $\beta_j(x)$ are easily seen to be Baire functions of the $G_0(x), \cdots, G_n(x)$; therefore, they are measurable.

For $\varphi(t, x)$ we now have the (redundant) expression

$$\varphi(t, x) = \sum_{\rho=0}^{n-1} \sum_{h=-n}^{n} \sum_{j=1}^{n} a_{\rho,h,j}(x)\, t^\rho e^{(\gamma_h(x)+i\beta_j(x))t}. \tag{43}$$

We wish to prove that all $a_{\rho,h,j}(x)$ are measurable. If this were not true, there would be a non-measurable $a_{\rho,h,j}(x)$ with a maximal h, say $h = \bar{h}$, and among those with $h = \bar{h}$ one with a maximal ρ, say $\rho = \bar{\rho}$. Let its j be $= \bar{j}$. Then we have

$$\varphi(t, x) - \sum_{\rho=0}^{n-1} \sum_{h=\bar{h}+1}^{n} \sum_{j=1}^{n} - \sum_{\rho=\bar{\rho}+1}^{n-1} \sum_{(h=\bar{h})}^{n} \sum_{j=1}^{n}$$

$$= \sum_{j=1}^{n} a_{\bar{\rho},\bar{h},j}(x)\, t^{\bar{\rho}} e^{(\gamma_{\bar{h}}(x)+i\beta_j(x))t} + \sum_{\rho=0}^{\bar{\rho}-1} \sum_{(h=\bar{h})} \sum_{j=1}^{n} + \sum_{\rho=0}^{n-1} \sum_{h=-n}^{\bar{h}-1} \sum_{j=1}^{n},$$

and a simple computation gives

$$\lim_{t'\to+\infty} \left[\frac{1}{t'} \int_0^{t'} \frac{1}{t^{\bar{\rho}}} \left(\varphi(t, x) - \sum_{\rho=0}^{n-1} \sum_{h=\bar{h}+1}^{n} \sum_{j=1}^{n} - \sum_{\rho=\bar{\rho}+1}^{n-1} \sum_{(h=\bar{h})} \sum_{j=1}^{n}\right) e^{-(\gamma_{\bar{h}}(x)+i\beta_j(x))t}\, dt\right] \tag{44}$$

$$= a_{\bar{\rho},\bar{h},\bar{j}}(x).$$

As the left side contains only $a_{\rho,h,j}(x)$'s with $h > \bar{h}$ or $h = \bar{h}$, $\rho > \bar{\rho}$, it is measurable, and thus $a_{\bar{\rho},\bar{h},j}(x)$ must be measurable too, contradicting our assumption. Thus all $a_{\rho,h,j}(x)$ are measurable.

We now see that both sides of (43) are (t, x)-measurable, and therefore the exceptional (t, x)-set in which (43) fails to hold is likewise (t, x)-measurable. But the exceptional set is for each x a t-set of measure 0, and hence, by Fubini's theorem (see [7], p. 627, theorem 1), it has a (t, x)-measure 0. Consequently, the exceptional points consist of a t-set $\overline{T}$ of measure 0, and for each t not in $\overline{T}$ of an x-set of measure zero (depending on t). As $\varphi(t, x)$ is defined only up to such x-sets we may assume that all exceptions to (43) are contained in a t-set $\overline{T}$ of measure 0.

Denote by $\epsilon(x)$, for each x, the smallest of all differences of any two $\gamma_h(x)$ and of any two $\beta_j(x)$. Let $\bar{n}(x) = 1, 2, \cdots$ be the smallest integer with $\bar{n}(x) > \frac{5}{\epsilon(x)}$, and $\bar{\gamma}_h(x)$ and $\bar{\beta}_j(x)$ the uniquely determined rational numbers with the denominator $\bar{n}(x)$ for which

$$\gamma_h(x) \leqq \bar{\gamma}_h(x) < \gamma_h(x) + \frac{1}{\bar{n}(x)}$$

$$\beta_j(x) \leqq \bar{\beta}_j(x) < \beta_j(x) + \frac{1}{\bar{n}(x)}.$$

Obviously $\epsilon(x)$, $\bar{n}(x)$, $\bar{\gamma}_h(x)$, $\bar{\beta}_j(x)$ are measurable, $\bar{\gamma}_0(x) = 0$, and

$$(45)\qquad \begin{aligned} &|\gamma_h(x) - \bar{\gamma}_h(x)| \leqq \frac{1}{\bar{n}(x)}, && |\beta_j(x) - \bar{\beta}_j(x)| \leqq \frac{1}{\bar{n}(x)} \\ &\bar{\gamma}_{h+1}(x) > \bar{\gamma}_h(x) + \frac{4}{\bar{n}(x)}, && \bar{\beta}_{j+1}(x) > \bar{\beta}_j(x) + \frac{4}{\bar{n}(x)}. \end{aligned}$$

We now consider all possible combinations of an $\bar{n} = 1, 2, \cdots$ and $3n + 1$ rational numbers $\bar{\gamma}_h$, $\bar{\beta}_j$ $(h = -\bar{n}, \cdots, \bar{n}; j = 1, \cdots, \bar{n})$ satisfying

$$(46)\qquad \bar{\gamma}_{h+1} > \bar{\gamma}_h + \frac{4}{\bar{n}}, \qquad \bar{\beta}_{j+1} > \bar{\beta}_j + \frac{4}{\bar{n}}, \qquad \bar{\gamma}_0 = 0.$$

They are countably many, and thus we may denote them by $C_1, C_2, \cdots$. Denote by $M_{N,P}$ the set of all x for which $\bar{n}(x)$, $\bar{\gamma}_h(x)$, $\bar{\beta}_j(x)$ coincide with the $\bar{n}$, $\bar{\gamma}_h$, β_j of C_N, and, besides,

$$P - 1 \leqq \text{l.u.b.}_{\rho,h,j} |a_{\rho,h,j}(x)| < P \qquad (P = 1, 2, \cdots).$$

The sets $M_{N,P}$ are measurable, mutually exclusive, and their sum is M. In each $M_{N,P}$ all $\bar{n}(x)$, $\bar{\beta}_h(x)$, $\bar{\gamma}_j(x)$ are constant, and $|a_{\rho,h,j}(x)| < P$.

Omitting those $M_{N,P}$ which have measure zero (§7), lemma 6 leads to the following

Specialization 5. We may assume in Specialization 4 that (except perhaps for a t-set $\overline{T}$ of measure 0)

$$\varphi(t,x)=\sum_{\rho=0}^{n-1}\sum_{h=-n}^{n}\sum_{j=1}^{n}a_{\rho,h,j}(x)\,t^{\rho}\,e^{(\gamma_h(x)+i\beta_j(x))t} \tag{43}$$

where $a_{\rho,h,j}(x)$, $\gamma_h(x)$, $\beta_j(x)$ are measurable functions, $\gamma_h(x)$, $\beta_j(x)$ are real, and a set of real constants $P>0$, $\bar{\delta}>0$, $\bar{\gamma}_h$, $\bar{\beta}_j$ exists, such that

$$|a_{\rho,h,j}(x)|<P, \tag{47}$$

$$|\gamma_h(x)-\bar{\gamma}_h|<\bar{\delta}, \qquad |\beta_j(x)-\bar{\beta}_j|<\bar{\delta}, \tag{45'}$$

$$\bar{\gamma}_{h+1}>\bar{\gamma}_h+4\bar{\delta}, \qquad \bar{\beta}_{j+1}>\bar{\beta}_j+4\bar{\delta}. \tag{46'}$$

Besides $\gamma_0(x)=\bar{\gamma}_0=0$.

10. With Specialization 5 the form of $\varphi(t,x)$ is sufficiently restricted to enable us to discuss directly the consequences of the compactness of φ_t (which has scarcely been used so far).

A consequence of the compactness is the boundedness of φ_t, and first we will use only this. We prove successively the following lemmas:

LEMMA 7. *Let $a_j(x)$, $\beta_j(x)$ be measurable functions, $|a_j(x)|<P$, $\beta_j(x)$ as in Specialization* 5. *Then*

$$\psi_t=\psi_t(x)=\psi(t,x)=\sum_{j=1}^{n}a_j(x)\,e^{i\beta_j(x)t} \tag{48}$$

belongs to $\mathfrak{H}$, and if not all $a_j(x)=0$ (except perhaps for an x-set of measure 0), *then* $\limsup_{t\to+\infty}\|\psi_t\|$ *is finite and* >0.

Proof. As $|\psi_t(x)|\leqq nP$, and M has a finite measure, ψ_t belongs to $\mathfrak{H}$ and ψ_t is bounded. So we have only to prove that $\lim_{t\to+\infty}\|\psi_t\|=0$ would imply $\sum_{j=1}^{n}\int_M|a_j(x)|^2\,dx=0$. $\lim_{t\to+\infty}\|\psi_t\|=0$ implies $\lim_{t'\to+\infty}\frac{1}{t'}\int_0^{t'}\|\psi_t\|^2\,dt=0$ but

$$\frac{1}{t'}\int_0^{t'}\|\psi_t\|^2\,dt=\frac{1}{t'}\int_0^{t'}\int_M|\psi(t,x)|^2\,dt\,dx$$

$$=\frac{1}{t'}\sum_{j,j'=1}^{n}\int_0^{t'}\int_M a_j(x)\,\overline{a_{j'}(x)}\,e^{i(\beta_j(x)-\beta_{j'}(x))t}\,dt\,dx$$

$$=\sum_{j=1}^{n}\int_M|a_j(x)|^2\,dx+\sum_{j,j'=1,j\neq j'}^{n}\frac{1}{t'}\int_M a_{j'}(x)\,\overline{a_{j'}(x)}\,\frac{e^{i(\beta_j(x)-\beta_{j'}(x))t'}-1}{i(\beta_j(x)-\beta_{j'}(x))}\,dx.$$

Denoting the measure of M by C, each term of the second Σ is majorized by $\frac{1}{t'} \cdot C \cdot P^2 \cdot \frac{1}{\delta}$, and hence tends to zero for $t' \to +\infty$. Therefore

$$\lim_{t' \to +\infty} \frac{1}{t'} \int_0^{t'} \| \psi_t \|^2 \, dt = \sum_{j=1}^{n} \int_M | a_j(x) |^2 \, dx ,$$

and this proves our statement.

LEMMA 8. *Let $a_{\rho,j}(x)$, $\beta_j(x)$ be measurable functions, $| a_{\rho,j}(x) | < P$, $\beta_j(x)$ as in Specialization 5. Then*

$$\chi_t = \chi_t(x) = \chi(t, x) = \sum_{\rho=0}^{n-1} \sum_{j=1}^{n} a_{\rho,j}(x) \, t^\rho \, e^{i\beta_j(x)t} \tag{49}$$

belongs to $\mathfrak{H}$, and if the greatest ρ for which not all $a_{\rho,j}(x) = 0$ (except perhaps for an x-set of measure 0) is $\rho = \bar{\rho}$, then $\limsup_{t \to +\infty} \frac{\| \chi_t \|}{t^{\bar{\rho}}}$ is finite and > 0.

Proof. We have

$$\chi_t = \sum_{\rho=0}^{\bar{\rho}} t^\rho \, \psi_{\rho,t}$$

where

$$\psi_{\rho,t} = \psi_{\rho,t}(x) = \psi_\rho(t, x) = \sum_{j=1}^{n} a_{\rho,j}(x) \, e^{i\beta_j(x)t} .$$

Thus

$$\frac{\chi_t}{t^{\bar{\rho}}} = \psi_{\bar{\rho},t} + \frac{\psi_{\bar{\rho}-1,t}}{t} + \cdots + \frac{\psi_{0,t}}{t^{\bar{\rho}}} .$$

By Lemma 7, $\limsup_{t \to +\infty} \left\| \frac{\psi_{\bar{\rho}-\rho,t}}{t^\rho} \right\| = 0$ for $\rho = 1, \cdots, \bar{\rho}$, and $\limsup_{t \to +\infty} \| \psi_{\bar{\rho},t} \|$ is finite and > 0. Hence the assertion of lemma 8.

LEMMA 9. *Let $a_{\rho,h,j}(x)$, $\gamma_h(x)$, $\beta_j(x)$ be measurable functions, all as in Specialization 5. Define $\omega(t, x)$ by*

$$\omega_t = \omega_t(x) = \omega(t, x) = \sum_{\rho=0}^{n-1} \sum_{h=-n}^{n} \sum_{j=1}^{n} a_{\rho,h,j}(x) \, t^\rho \, e^{(\gamma_h(x)+i\beta_j(x))t} . \tag{50}$$

It belongs to $\mathfrak{H}$, and if the greatest h for which not all $a_{\rho,h,j}(x) = 0$ (except perhaps for an x-set of measure 0) is $h = \bar{h}$ and $\bar{h} = 1, \cdots, n$, then $\limsup_{t \to +\infty} \| \omega_t \| = +\infty$.

Proof. Write

$$\omega_t = \omega_t(x) = \omega(t, x) = \sum_{h=-n}^{\bar{h}} e^{\gamma_h(x)t} \, \chi_h(t, x) ,$$

$$\chi_{h,t} = \chi_{h,t}(x) = \chi_h(t, x) = \sum_{\rho=0}^{n-1} \sum_{j=1}^{n} a_{\rho,h,j}(x) \, t^\rho \, e^{i\beta_j(x)t} .$$

Now by $\gamma_{\bar{h}}(x) > \bar{\gamma}_{\bar{h}} - \bar{\delta} > \bar{\gamma}_{\bar{h}-1} + 3\bar{\delta}$, $\gamma_h(x) < \bar{\gamma}_h + \bar{\delta} \leqq \bar{\gamma}_{\bar{h}-1} + \bar{\delta}$

$$|| \omega_t || \geqq e^{(\bar{\gamma}_{\bar{h}-1}+3\bar{\delta})t} || \chi_{\bar{h},t} || - \sum_{h=-n}^{\bar{h}-1} e^{(\bar{\gamma}_{\bar{h}-1}+\bar{\delta})t} || \chi_{h,t} ||,$$

$$e^{-(\bar{\gamma}_{\bar{h}-1}+2\bar{\delta})t} || \omega_t || \geqq e^{\bar{\delta}t} || \chi_{\bar{h},t} || - e^{-\bar{\delta}t} \sum_{h=-n}^{\bar{h}-1} || \chi_{h,t} ||. \tag{51}$$

As not all $a_{\rho,\bar{h},j}(x) = 0$ (except perhaps for an x-set of measure 0), Lemma 8 applies to the first term on the right of (51) and thus its $\limsup_{t\to+\infty}$ is $= +\infty$. As obviously every $|| \chi_{h,t} ||$ is $O(t^{n-1})$ for $t \to +\infty$, the other terms on the right of (51) tend to 0 for $t \to +\infty$. Thus $\limsup_{t\to+\infty} e^{-(\bar{\gamma}_{\bar{h}-1}+2\bar{\delta})t} || \omega_t || = +\infty$, and a fortiori $\limsup_{t\to+\infty} || \omega_t || = +\infty$.

If ω_t, as defined by (50), has $\limsup_{t\to+\infty} || \omega_t || = \infty$, then, owing to its continuity in t, $|| \omega_t ||$ would become arbitrarily large, for $t \to +\infty$, *in whole intervals*. But the solution φ_t, see (43), equals ω_t, except for a t-set of measure. $|| \varphi_t ||$ being bounded, we conclude from lemma 9 that, for $h \geqq 1$, $a_{\rho,h,j}(x)$ vanishes except for an x-set of measure 0. By §7, we may assume that the sum (43) does not actually contain any term with $h \geqq 1$. Replacing t by $-t$ transforms all $\gamma_h(x)$ into $-\gamma_h(x)$, and using the boundedness of $|| \varphi_t ||$ for $t \to -\infty$ we conclude that (43) does not actually contain any term also for $h \leqq -1$. Thus φ_t is, up to a t-set of measure 0, a function χ_t, as defined by (49). Applying lemma 8, we conclude in the same way, that χ_t is, up to a t-set of measure 0, an expression of the form ψ_t, see (48). Thus we have

$$\varphi_t = \varphi_t(x) = \varphi(t, x) = \sum_{j=1}^{n} a_j(x)\, e^{i\beta_j(x)t}, \tag{52}$$

except for a t-set $\overline{T}$ of measure 0; (compare the discussion concerning the relation (43) in §9). In (52), the $\beta_j(x)$ and $a_j(x)$ are the $\beta_j(x)$ and $a_{0,0,j}(x)$ of specialization 5).

11. From now on we are going to use the full strength of the decisive assumption that φ_t is compact.

First, remark that if we define

$$\omega_t = \omega_t(x) = \omega(t, x) = \sum_{j=1}^{n} a_j(x)\, e^{i\beta_j(x)t} \tag{53}$$

without t-exceptions, $\varphi_t = \omega_t$ except on t-set of measure 0. Thus $\varphi_t = \omega_t$ on a dense t-set, and as the φ_t of this t-set also are compact, ω_t is compact on a dense t-set. But as ω_t is continuous in t, all ω_t are limits of these ω_t—and thus the set of all ω_t is compact, too.

A simple computation shows that ω_t is even uniformly continuous in t.

Next we prove:

LEMMA 10. *Let v_t be a variable point of $\mathfrak{H}$, uniformly continuous in t, the set of all v_t compact. Let $w(t)$ be a complex-valued function of t, with a finite* $\int_{-\infty}^{+\infty} |w(t)|\,dt$. *Then the set of all*

$$\eta_s = \int_{-\infty}^{+\infty} w(t)\, v_{t+s}\, dt$$

is also compact.

Proof. As the set v_s is compact, each sequence $v_{s1}, v_{s2}, \cdots$ has a convergent subsequence.

If a sequence $t_1, t_2, \cdots$ is given, the well-known diagonal-process allows us to select from any sequence $s_1, s_2, \cdots$ a subsequence $\sigma_1, \sigma_2, \cdots$, such that the sequence

$$v_{t+\sigma_1}, \quad v_{t+\sigma_2}, \cdots \tag{54}$$

converges for every $t = t_k$ $(k = 1, 2, \cdots)$. If the $t_1, t_2, \cdots$ are chosen dense, the uniform continuity of v_t implies that (54) converges for every t, even uniformly. Hence, applying the hypotheses of the lemma, the sequence $\eta_{\sigma_1}, \eta_{\sigma_2}, \cdots$ is convergent. Thus the set of all η_s is compact.

There exists a numerical function $w_\delta(t)$, for which $\int_{-\infty}^{+\infty} |w_\delta(t)|\,dt$ is finite, and the Fourier-transform of which,

$$W_\delta(\beta) = \int_{-\infty}^{+\infty} w_\delta(t)\, e^{i\beta t}\, dt$$

has these values:

$$W_\delta(\beta) \begin{cases} = 1 & \text{for } |\beta| \leqq \delta \\ = 2 - \dfrac{|\beta|}{\delta} & \text{for } \delta \leqq |\beta| < 2\delta \\ = 0 & \text{for } |\beta| \geqq 2\delta \end{cases}$$ [29]

Putting in Lemma 10, $v_t = \omega_t$, $w(t) = w_{\bar\delta}(t)\, e^{-i\bar\beta_j t}$, we obtain (comp. §3, Part I)

$$\eta_s(x) = \int_{-\infty}^{+\infty} w(t)\, \omega(t, x)\, dt = \sum_{j'=1}^{n} a_{j'}(x)\, e^{i\beta_{j'}(x)t} \int_{-\infty}^{+\infty} w_{\bar\delta}(t)\, e^{i(\beta_{j'}(x)-\bar\beta_j)t}\, dt$$

$$= a_j(x)\, e^{i\beta_j(x)t}.$$

Therefore Lemma 10 states:

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[29] Cf. [5], p. 89, 90. This function is

$$w_\delta(t) = \frac{\cos(\delta t) - \cos(2\delta t)}{\pi \delta t^2}.$$

For each $j = 1, 2, \cdots, n$ the set

$$\eta_t = \eta_t(x) = \eta(t, x) = a_j(x)\, e^{i\beta_j(x)t}$$

is compact.

12. LEMMA 11. *Let $\alpha(x)$, $\beta(x)$ be bounded and measurable functions, $\beta(x)$ real, and let the set*

$$\eta_t = \eta_t(x) = \eta(t, x) = a(x)\, e^{i\beta(x)t} \tag{55}$$

be compact.

There exists a (finite or infinite) sequence of mutually exclusive measurable sets $M^*, M_1, M_2, \cdots, M^* + M_1 + M_2 + \cdots = M$, such that $\beta(x)$ is constant in each M, and $a(x) = 0$ in M^* except for a set of measure 0.

Proof. Denote, for each β, the set of all x with $\beta(x) = \beta$ by $M(\beta)$. As the $M(\beta)$ are mutually exclusive, only a finite or countable number of them can have a positive measure. Call these $\beta : \beta_1, \beta_2, \cdots$, their $M(\beta) : M_1, M_2 \cdots$, and put $M^* = M - M_1 - M_2 - \cdots$. All we have to prove, is $\int_{M^*} |\, a(x)\,|^2\, dx = 0$, that is $\int_{M^*} |\, \eta_t(x)\,|^2\, dx = 0$.

Define

$$\left\{\begin{aligned} \int_{M^*} |\, \eta_t(x) - \eta_s(x)\,|^2\, dx &= \int_{M^*} |\, a(x)\,|^2\, |\, e^{i\beta(x)t} - e^{i\beta(x)s}\,|^2\, dx \\ &= 4 \int_{M^*} |\, a(x)\,|^2\, (\sin \tfrac{1}{2}\beta(x)(t-s))^2\, dx = F(t-s)\,. \end{aligned}\right. \tag{56}$$

The well-known "triangle relation" gives $\sqrt{F(t-r)} \leqq \sqrt{F(t-s)} + \sqrt{F(s-r)}$. Putting $r = 0$, and considering both pairs t, s and s, t leads to $|\, F(t) - F(s)\,| \leqq |\, F(t-s)\,|$, and this implies (viewing $F(0) = 0$)

$$\text{l.u.b.}_r\, |\, F(r+t) - F(r)\,| = F(t)\,. \tag{57}$$

By (56) we have

$$F(t-s) \leqq \|\, \eta_t - \eta_s\,\| \tag{58}$$

and combining (57), (58):

$$\text{l.u.b.}_r\, |\, F(r+t) - F(r+s)\,| \leqq \|\, \eta_t - \eta_s\,\|. \tag{59}$$

Thus the compactness of the set of the η_t implies the compactness of the set of the r-functions $F(r+t)$ (the distance of $G(r)$ and $H(r)$ being defined by least upper bound $|\, G(r) - H(r)\,|$). This means that $F(r)$ is almost periodic[30] and thus $F(r)$ too.

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[30] For Bohr's notion of almost periodicity and its properties, see [6]. The form of definition which appears above is due to Bochner, see [1], p. 143.

The function

$$\rho(\beta) = \int_{S_\beta} |\, a(x)\,|^2\, dx, \qquad S_\beta = \text{set of all } x\text{'s with } \beta(x) \leqq \beta,\ x \text{ in } M^*,$$

is non-decreasing. If $|\, \beta(x)\,| \leqq P$ everywhere, we have

$$\rho(\beta) = \begin{cases} \text{constant} = 0, & \text{for} \quad \beta < -P \\ \text{constant} = \int_{M^*} |\, a(x)\,|^2\, dx = \mu_0, & \text{for} \quad \beta \geqq +P. \end{cases}$$

Using $\rho(\beta)$, (56) goes over into

$$F(r) = 4 \int_{-P}^{P} (\sin \tfrac{1}{2}\beta r)^2\, d\rho(\beta) \tag{60}$$

(Stieltjes-integral!), and the "distribution function" $\rho(\beta)$ has no discontinuities. The latter follows by our construction of M^*: any set of M^* on which $\beta(x)$ has a fixed value β_0 has the measure 0. Putting $G(r) = 2\mu_0 - F(r)$, $\sigma(\beta) = \rho(\beta) - \rho(-\beta)$, (60) reads

$$G(r) = \int_{-P}^{P} e^{i\beta r}\, d\sigma(\beta)$$

and, hence, owing to the continuity of $\sigma(\beta)$,

$$\lim_{t\to\infty} \frac{1}{2t} \int_{-t}^{t} |\, G(r)\,|^2\, dr = 0\,^{31}. \tag{61}$$

As $G(r)$ is almost periodic, this implies $G(r) = 0$, thus $F(r) = 2\mu_0$. But $F(0) = 0$; thus $\mu_0 = 0$, or $\int_{M^*} |\, a(x)\,|^2\, dx = 0$.

13. Now we apply lemma 11 to each pair $a_j(x)$, $\beta_j(x)$ appearing in (53), thus introducing sets M_j^*, $M_{j,1}$, $M_{j,2}$, $\cdots$. In $M_{j,k}$, $\beta_j(x)$ is constant; in M_j^* we may also give it any constant value, since $a_j(x)$ is zero in M_j^* except for an x-set of measure zero (which may be ignored, §7); and we denote M_j^* by $M_{j,0}$. Then $\beta_j(x)$ is a constant in each $M_{j,N}$, $N = 0, 1, 2, \cdots$. Taking the intersection of any combination of n sets M_{j,N_j} $(j = 1, \cdots, n)$, we obtain a sequence of mutually exclusive measurable sets $M_1, M_2, \cdots$, with $M_1 + M_2 + \cdots = M$, such that every $\beta_j(x)$ is constant in every M_N. Omitting the M_N of measure 0, and applying lemma 6, we obtain

Specialization 6. We may assume in Specialization 4 that

$$\varphi_t = \varphi_t(x) = \varphi(t, x) = \sum_{j=1}^{l} a_j(x)\, e^{i\beta_j t} \qquad (l \leqq n) \tag{62}$$

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[31] See, e.g., [9], p. 257, 258, formula (2), or [5], p. 80, formula (13).

(except for a t-set of measure zero), where β_j is a constant, and $a_j(x)$ is measurable and bounded.

If some $a_j(x)$ are 0 except for x-sets of measure 0, we may omit them from the sum in (62), thus we may assume $\int |a_j(x)|^2\,dx > 0, j = 1, 2, \cdots, l$. Defining an element ψ_j of $\mathfrak{H}(= L_2^{(M)})$ by $\psi_j = \psi_j(x) = a_j(x)$ we thus have

$$\varphi_t = \sum_{j=1}^{l} e^{i\beta_j t}\,\psi_j \qquad \text{all } \psi_j \neq 0\,; \tag{63}$$

and, by Specialization 5, all β_j are different.

We substitute (63) in (26) and differentiate n times (compare the discussion concerning relation (40) in §9). This gives

$$\sum_{\nu=0}^{n} (i\beta_j)^{n-\nu} A_\nu \psi_j = 0\,. \tag{64}$$

Now it is clear that with $\mathfrak{M}_1 = \mathfrak{H}$ (no other $\mathfrak{M}_N$!), $\varphi_t^1 = \varphi_t$, $l_1 = l$, $\beta_{1,j} = \beta_j$, $\psi_{1,j} = \psi_j$ all statements $A - E$ of theorem I are fulfilled.

Part IV. Proof of Theorem III

1. We shall assume, *throughout this part*, without further mention, that the operators $A_0, \cdots, A_n, A$ and the function φ_t satisfy the hypothesis of theorem III. We shall treat Theorem III in various steps, each imposing an additional restriction on the operator A appearing in the hypothesis of the theorem. This restriction will become weaker as we proceed, and in the end will disappear completely.

2. Let $\mathfrak{N}$ be any closed linear manifold of $\mathfrak{H}$, which reduces A. Obviously the hypotheses of Theorem III are also true in $\mathfrak{N}$, that is for the part of A in $\mathfrak{N}$, the same polynomials (31), and the projection of φ_t in $\mathfrak{N}$. This enables us to formulate the following lemmas.

LEMMA 12. *Let $\mathfrak{N}$ be a closed linear manifold which reduces A. If Theorem III holds in $\mathfrak{H}$ it also holds in $\mathfrak{N}$.*

Proof. We form the common part $\overline{\mathfrak{M}}$ of $\mathfrak{N}$ with the manifold $\mathfrak{M}$ for which Theorem III holds true in $\mathfrak{H}$. Then the assertions $A' - C'$ hold true for $\overline{\mathfrak{M}}$ in $\mathfrak{N}$.

LEMMA 13. *Let $\mathfrak{N}_1, \mathfrak{N}_2, \cdots$ be a (finite or infinite) sequence of mutually orthogonal closed linear manifolds, with $\mathfrak{N}_1 + \mathfrak{N}_2 + \cdots = \mathfrak{H}$, each $\mathfrak{N}_N$ reducing A. If the statements $A' - C'$ of Theorem III hold for each $\mathfrak{N}_N$, then they also hold in the entire $\mathfrak{H}$.*

Proof. Denoting by $\mathfrak{M}_N$ the $\mathfrak{M}$ for which Theorem III holds in $\mathfrak{N}_N$, it is clear that $\mathfrak{M} = \mathfrak{M}_1 + \mathfrak{M}_2 + \cdots$ fulfills $A' - C'$ in $\mathfrak{H}$.

3. We now make the additional restriction that A be hypermaximal. If, moreover, assumption *d)*, see §1, part II, were fulfilled, our theorem would be a straightforward consequence of theorem I[6]—but assumption *d)* was explicitly

exempted in Theorem III. Thus we have to concern ourselves with those elements ψ which violate assumption d), that is with those elements $\psi \neq 0$, for which, simultaneously,

$$\sum_{\lambda=0}^{m} a_{\nu,\lambda} A^{\lambda} \psi = 0, \qquad \nu = 0, 1, \cdots, n. \tag{65}$$

We shall prove two lemmas.

LEMMA 14. *Let A be a (closed) Hermitian operator, and $\mathfrak{C}$ a set of real numbers. Let ψ be any element of $\mathfrak{H}$, $\neq 0$, such that for some c in $\mathfrak{C}$*

$$A\psi = c\psi, \tag{66}$$

and let $\mathfrak{F}_{\mathfrak{C}}$ denote the set of all these elements ψ (that is of all proper functions of A corresponding to any point-proper-value belonging to $\mathfrak{C}$).

Then $\mathfrak{F}_{\mathfrak{C}}$ determines a closed linear manifold $\mathfrak{L}_{\mathfrak{C}}$ which reduces A; and A in $\mathfrak{L}_{\mathfrak{C}}$ has a pure point spectrum, all its proper values belonging to $\mathfrak{C}$.

Proof. If ψ satisfies (66) for some c in $\mathfrak{C}$, then $[\psi]$ obviously reduces A. If $\mathfrak{C}$ consists of c alone, $\mathfrak{C} = \{c\}$, then the solutions ψ of (66) form a closed linear manifold, that is $\mathfrak{F}_{\{c\}} = \mathfrak{L}_{\{c\}}$; and if $\psi_1, \psi_2, \cdots$ is a maximal normalized orthogonal system of these solutions, then $\mathfrak{L}_{\{c\}} = [\psi_1] + [\psi_2] + \cdots$, and therefore $\mathfrak{L}_{\{c\}}$ also reduces A.[32]

For $c \neq d$, $\mathfrak{L}_{\{c\}}$ and $\mathfrak{L}_{\{d\}}$ are orthogonal (because $A\psi = c\psi$, $A\varphi = d\varphi$, $c \neq d$ imply $(A\psi, \varphi) = (\varphi, A\psi)$, $c(\psi, \varphi) = d(\psi, \varphi)$, $(\psi, \varphi) = 0$); hence $\mathfrak{L}_{\{c\}} \neq 0$ is possible only for a countable number of c's. Thus $\mathfrak{L}_{\mathfrak{C}}$is the sum of a countable number of mutually orthogonal sets $\mathfrak{L}_{\{c\}}$. Since each of these sets reduces A, their sum also does, and it is obvious that A in $\mathfrak{L}_{\mathfrak{C}}$ has a pure point spectrum consisting of those numbers c which appeared in the argument.

LEMMA 15. *Given a (close) Hermitian operator A, and a system of polynomials*

$$\sum_{\lambda=0}^{m} a_{\nu,\lambda} z^{\lambda}, \qquad \nu = 0, 1, \cdots, n, \tag{67}$$

not all of which are identically zero, the set of elements ψ satisfying

$$\sum_{\lambda=0}^{m} a_{\nu,\lambda} A^{\lambda} \psi = 0, \qquad \nu = 0, 1, \cdots, n, \tag{65}$$

is contained in the closed linear manifold determined by all point-proper-functions ψ of A satisfying (65), *that is in the manifold $\mathfrak{L}_{\mathfrak{C}}$ of lemma* 14, *where $\mathfrak{C}$ is the set of all z satisfying*

$$\sum_{\lambda=0}^{m} a_{\nu,\lambda} z^{\lambda} = 0, \qquad \nu = 0, 1, \cdots, n. \tag{68}$$

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[32] If $\mathfrak{M}_1, \mathfrak{M}_2, \cdots$ are mutually orthogonal, and each reduces A, then $\mathfrak{M} = \mathfrak{M}_1 + \mathfrak{M}_2 + \cdots$ also reduces A; the reason being that $P_{\mathfrak{M}} = P_{\mathfrak{M}_1} + P_{\mathfrak{M}_2} + \cdots$ and that $\mathfrak{M}$ reduces A if and only if $P_{\mathfrak{M}}$ commutes with A (compare, for instance, [11], p. 79, theorem 20).

Proof. If all polynomials (67) are constants, since at least one is $\not\equiv 0$, we have an $a\,\psi = 0$, $a \neq 0$. Hence only $\psi = 0$ satisfies (65) and the lemma holds. If not all polynomials (67) are constants we can find one of the form

$$\sum_{\lambda=0}^{k} b_\lambda z^\lambda \qquad 1 \leqq k \leqq m, \qquad b_k \neq 0.$$

We may assume $b_k = -1$ $\left(\text{otherwise we multiply by } -\frac{1}{b_k}\right)$. Therefore we have

$$A^k\psi = \sum_{\lambda=0}^{k-1} b_\lambda A^\lambda \psi \tag{69}$$

or all ψ's in question.

Given ψ, let $\mathfrak{M}$ be the closed linear manifold determined by $\psi, A\psi, \cdots, A^{k-1}\psi$. ($\mathfrak{M}$ depends on ψ!.) By (69) $\mathfrak{M}$ contains $A^k\psi$, that is, it contains $A\psi, A^2\psi, \cdots, A^k\psi$. Hence, if $\mathfrak{M}$ contains f it also contains Af, which means that $\mathfrak{M}$ reduces f (comp. [11], p. 78–80, and [22], p. 150–151). As $\mathfrak{M}$ is finite dimensional, A has a pure point spectrum in $\mathfrak{M}$, which implies that there exists a complete normalized orthogonal system $\psi_1, \cdots, \psi_k$ in $\mathfrak{M}$ consisting of point-proper-functions of A.—Since (65) is satisfied for ψ, it is also satisfied for $A\psi, \cdots, A^{k-1}\psi$, and, in consequence, for all elements of $\mathfrak{M}$—in particular for $\psi_1, \cdots, \psi_k$. Thus ψ really belongs to the closed linear manifold determined by all point-proper-functions of A satisfying (65), and this proves the first statement of our lemma.

As to the second statement of our lemma, if a point-proper-function ψ of A, $\psi \neq 0$, pertaining to the proper-value z (that is a solution of $A\psi = z\psi$) satisfies (65), then z must obviously satisfy (68).

4. Now we are in a position to prove Theorem III for a hypermaximal operator A.

Let $\mathfrak{C}_0$ be the set of all z satisfying (68). By lemma 14, $\mathfrak{L}_{\mathfrak{C}_0}$ reduces A, and contains all solutions ψ of (65).—Consider now the situation in $\mathfrak{H} - \mathfrak{L}_{\mathfrak{C}_0}$. It also reduces A, and being orthogonal to $\mathfrak{L}_{\mathfrak{C}_0}$ it contains by lemma 15 no element $\psi \neq 0$ which satisfies (65). Hence assumption *d*), see §1, part II, is fulfilled and (as we remarked at the beginning of the preceding §3) Theorem I applies to A in $\mathfrak{H} - \mathfrak{L}_{\mathfrak{C}_0}$ (remember Lemma 12!).

By (30), (31) we have

$$\sum_{\lambda=0}^{m} \left(\sum_{\nu=0}^{n} a_{\nu,\lambda} (i\beta_{N,q})^{n-\nu} \right) A^\lambda \psi_{N,q} = 0, \tag{70}$$

and thus the situation of lemma 15 arises, the family of polynomials (67) being replaced by the one polynomial

$$\begin{cases} \sum_{\lambda=0}^{m} \alpha_\lambda z^\lambda, \\ \alpha_\lambda = \sum_{\nu=0}^{n} a_{\nu,\lambda} (i\beta_{N,q})^{n-\nu}. \end{cases} \tag{71}$$

Since

$$\sum_{\lambda=0}^{m} \alpha_\lambda z^\lambda \equiv 0$$

was excluded in Theorem III, lemma 15 actually applies: if $\mathfrak{D}_{N,q}$ is the set of all solutions of

$$\sum_{\lambda=0}^{m} \alpha_\lambda z^\lambda = 0 \tag{72}$$

then $\psi_{N,q}$ belongs to $\mathfrak{L}_{\mathfrak{D}_{N,q}}$.

Now let R be the set of all real numbers, and $\mathfrak{L}_R$ the closed linear manifold determined by all point-proper-functions of A in $\mathfrak{H}$. $\mathfrak{L}_{\mathfrak{C}_0}$ is a subset of $\mathfrak{L}_R$, thus the projection of ψ_t into $\mathfrak{L}_{\mathfrak{C}_0}$ belongs to $\mathfrak{L}_R$. $\mathfrak{L}_{\mathfrak{D}_{N,q}}$ is also a subset of $\mathfrak{L}_R$, thus all $\psi_{N,q}$ belong to $\mathfrak{L}_R$. Therefore by (28), (29), all φ_t^N, and except perhaps for a t-set of measure 0, the projection of φ_t into $\mathfrak{H} - \mathfrak{L}_{\mathfrak{C}_0}$ belong to $\mathfrak{L}_R$. (Remember that Theorem I had been applied in $\mathfrak{H} - \mathfrak{L}_{\mathfrak{C}_0}$, not in $\mathfrak{H}$!.) Therefore, with the above t-exception, all φ_t belong to $\mathfrak{L}_R$. Furthermore, by lemma 14, $\mathfrak{L}_R$ reduces A, and A has a pure point spectrum in $\mathfrak{L}_R$. Of course, it is there also hypermaximal. Thus $\mathfrak{M} = \mathfrak{L}_R$ satisfies the condition $A'. - C'.$ in Theorem III. This completes the proof.

5. Next, let A be maximal only (compare [22], p. 50, and [11], p. 72). In this case there exists a (finite or infinite) sequence of mutually orthogonal closed linear manifolds $\mathfrak{N}, \mathfrak{M}_1, \mathfrak{M}_2, \cdots$, with $\mathfrak{N} + \mathfrak{M}_1 + \mathfrak{M}_2 + \cdots = \mathfrak{H}$, each reducing A, such that A in $\mathfrak{N}$ is hypermaximal and A in any $\mathfrak{M}_N$ is isomorphic to the "exceptional operator" $\bar{R}$ or to $-\bar{R}$ (compare [22], p. 351, theorem 9.10, and [11], p. 98–99, theorem 38). By lemma 13, it is sufficient to establish Theorem III in $\mathfrak{N}, \mathfrak{M}_1, \mathfrak{M}_2, \cdots$ separately. A in $\mathfrak{N}$ is hypermaximal, and this case has just been settled. As to A in $\mathfrak{M}_N$ we shall prove that the projection of φ_t in $\mathfrak{M}_N$ is 0; and this readily implies Theorem III, as we can choose $\mathfrak{M} = 0$.

We consider, as in [11], p. 130, theorem 2**, the Hilbert-space $L_2^{(0,1)}$ (denoted loc. cit. by $\mathfrak{H}^*$) of all complex-valued functions $f(x)$ in $0 \leqq x \leqq 1$ with a finite $\int_0^1 |f(x)|^2\,dx$ the inner product being $(f, g) = \int_0^1 f(x)\bar{g}(x)dx$; and in it the closed linear manifold $\mathfrak{J}_0^+$ determined by the elements $f_n = e^{2\pi i n x}$, $n = 0, 1, 2, \cdots$. Since $\mathfrak{J}_0^+$ is infinite dimensional it may, and will, be considered as a Hilbert space to itself. We furthermore consider in $L_2^{(0,1)}$ the operator

$$Rf(x) = -ctg(\pi x) \cdot f(x)\,. \tag{73}$$

If f belongs to $\mathfrak{J}_0^+$ and Rf has a sense, then Rf also belongs to $\mathfrak{J}_0^+$, and thus we can speak of the part of R in $\mathfrak{J}_0^+$, see loc. cit.[33] Now the "exceptional operator"

[33] Nevertheless, $\mathfrak{J}_0^+$ does not reduce R, because Rf may be defined, without Rf_1 being defined, where f_1 is the projection of f into $\mathfrak{J}_0^+$.

$\bar{R}$ in $\mathfrak{M}_N$ is isomorphic to the part of R in $\mathfrak{H}_0^+$, if $\mathfrak{H}_0^+$ is considered as a Hilbert space. Thus we have to prove $\varphi_t = 0$ in the realization $\mathfrak{H} = \mathfrak{H}_0^+, A = \pm R$. A can be replaced by $-A$, hence we may assume $A = R$. But now we shall investigate our proposition not in $\mathfrak{H}_0^+$ but in $L_2^{(0,1)}$ itself.

As the operator (73) is hypermaximal in $L_2^{(0,1)}$ (compare loc. cit.), Theorem III applies here. We conclude that (except for a t-set of measure 0) φ_t belongs to the closed linear manifold $\mathfrak{M}$ which is determined by the proper-functions of R in $L_2^{(0,1)}$. Now $R\omega = c\omega$ means $-ctg(\pi x)\omega(x) = c\omega(x)$, hence $\omega(x) = 0$. Therefore $\omega = 0, \varphi_t = 0$. This completes the proof.

6. For the sake of a subsequent application we observe that, for maximal A, we have proved Theorem III, and this has the following consequences. Let U be the Cayley-transform of A (compare [22], p. 335, and [11], p. 80). $\mathfrak{M}$ reduces also U; and U is unitary in $\mathfrak{M}$, because A in $\mathfrak{M}$ is hypermaximal (compare [22], p. 88, and [11], p. 351). Therefore all powers $U^q, q = 0, \pm 1, \pm 2, \cdots$ are defined throughout $\mathfrak{M}$, and, in particular all $U^q \varphi_t$ are defined (except for a t-set of measure 0).

7. Finally let A be an entirely arbitrary Hermitian operator. We may assume that its deficiency-indices m, n (compare [22], p. 81, 338, and [11], p. 87) are both infinite, which will be justified by the following construction.

Take a Hilbert space $\overline{\mathfrak{H}}$ and in it two infinite-dimensional closed linear manifolds, $\mathfrak{H}_1, \mathfrak{H}_2$, with $\mathfrak{H}_1 + \mathfrak{H}_2 = \overline{\mathfrak{H}}$, and identify $\mathfrak{H}$ with $\mathfrak{H}_1$. Choose any closed Hermitian operator A_2 in $\mathfrak{H}_2$ which has the deficiency indices ∞, ∞; and consider in $\overline{\mathfrak{H}}$ the (closed Hermitian) operator $\bar{A}$ which in $\mathfrak{H}_1 \equiv \mathfrak{H}$ coincides with $A_1 \equiv A$, and in $\mathfrak{H}_2$ with A_2. The deficiency indices of $\bar{A}$ are the sums of those of A_1 and A_2, that is ∞, ∞. And, since $\mathfrak{H}_1$ reduces $\bar{A}$, on the strength of lemma 12, it is sufficient to prove that Theorem III holds for $\bar{A}$ in $\overline{\mathfrak{H}}$. We shall write A and $\mathfrak{H}$ for $\bar{A}$ and $\overline{\mathfrak{H}}$.

Denote by U the Cayley-transform of A (compare loc. cit. §6), its definition domain by $\mathfrak{E}$, its range by $\mathfrak{F}$. The dimensionalities of $\mathfrak{H} - \mathfrak{E}, \mathfrak{H} - \mathfrak{F}$ are the deficiency-indices, hence both $= \infty$.

We will prove that (except for a t-set of measure 0) all $U^q \varphi_t, q = 0, \pm 1, \pm 2, \cdots$ have a sense. As we can replace A by $-A$, and thus U by U^{-1}, we may restrict ourselves to $q \geqq 0$. We shall proceed by induction. $q = 0$ is obvious; let us assume, therefore, that our statement holds for q but not for $q + 1$.

Let $\mathfrak{G}$ be the range of U^q, and $\omega_1, \omega_2, \cdots$ a (certainly infinite!) normalized orthogonal system which determines the closed linear manifold $\mathfrak{H} - \mathfrak{E}$; and let $\bar{\omega}_k$ be the $\mathfrak{G}$-projection of $\omega_k, k = 1, 2, \cdots$. By definition, $U^{q+1}\psi = U(U^q\psi)$. Therefore $U^{q+1}\psi$ has a sense if $U^q\psi$ lies simultaneously in $\mathfrak{G}$ and in $\mathfrak{E}$. The first means that ψ belongs to the U^{-q}-image of $\mathfrak{G}$, the second means that $U^q\psi$ is orthogonal to $\omega_1, \omega_2, \cdots$; or (in conjunction with the first) that $U^q\psi$ is orthog-

onal to $\bar{\omega}_1, \bar{\omega}_2, \cdots$; or, that ψ is orthogonal to all $U^{-q}\bar{\omega}_k$.[34] Hence $U^{q+1}\varphi_t$ exists if and only if φ_t is orthogonal to the U^{-q}-image of $\mathfrak{H} - \mathfrak{G}$, as well as to all $U^{-q}\bar{\omega}_k$. The t-set T in which φ_t fails to have these properties is measurable, and we have to refute the assumption that the measure of T is > 0. If this measure is > 0, then, for some $k = \bar{k}$, even the t-set $T_{\bar{k}}$, in which φ_t is not orthogonal to the U^{-q}-image of $\mathfrak{H} - \mathfrak{G}$ or to $U^{-q}\bar{\omega}_{\bar{k}}$ is of measure > 0. Now $T_{\bar{k}}$ consists of the points t, where either φ_t is not in the U^{-q}-image of $\mathfrak{G}$, or $(\varphi_t, U^{-q}\bar{\omega}_{\bar{k}}) = (U^q\varphi_t, \bar{\omega}_{\bar{k}}) = (U^q\varphi_t, \omega_{\bar{k}}) \neq 0$. φ_t fails to be in the U^{-q}-image of $\mathfrak{G}$ only if $U^q\varphi_t$ is not defined, but these points have measure 0. Hence: in a t-set of measure > 0, the element $U^q\varphi_t$ is defined and $(U^q\varphi_t, \omega_{\bar{k}}) \neq 0$. By a permutation of the $\omega_1, \omega_2, \cdots$ we can obtain $\bar{k} = 1$.

The closed linear manifold $[\omega_2, \omega_3, \cdots] = \mathfrak{H} - \mathfrak{E} - [\omega_1]$, (which is determined by $\omega_2, \omega_3, \cdots$), is infinite-dimensional, and therefore we can map it on $\mathfrak{H} - \mathfrak{F}$ by a length-preserving transformation V. Hence we can map $\mathfrak{E} + (\mathfrak{H} - \mathfrak{E} - [\omega_1]) = \mathfrak{H} - [\omega_1]$ on $\mathfrak{F} + (\mathfrak{H} - \mathfrak{F}) = \mathfrak{H}$ by a length-preserving transformation U', which is a continuation of both U and V. (See [22], p. 81, 82, 337, and [11], p. 83, 84.) Thus U' is the Cayley-transform of a Hermitian operator A', which is a continuation of A (loc. cit.), and maximal, since the range of its U' is $\mathfrak{H}$. In the t-set considered above $U^q\varphi_t$ has a sense, and, hence, also $U'^q\varphi_t = U^q\varphi_t$, but it is not orthogonal to ω_1, thus not in the definition domain of U'—that is, $U'^{q+1}\varphi_t$ has no sense.

But A' is maximal, and thus we have a contradiction to the result formulated in §6.

Denote by $\overline{T}_0$ the set of t-values for which not all $U^q\varphi_t$, $q = 0, \pm 1, \pm 2, \cdots$ have a sense. We have proved that $\overline{T}_0$ has measure 0. Denote by $\overline{\mathfrak{M}}$ the closed linear manifold determined by all $U^q\varphi_t$, $q = 0, \pm 1, \pm 2, \cdots$, t not in $\overline{T}_0$. U, U^{-1} are defined everywhere in $\overline{\mathfrak{M}}$ and map it on itself. Hence $\overline{\mathfrak{M}}$ reduces U, and therefore also A (comp. loc. cit. above). For the same reasons U is unitary in $\overline{\mathfrak{M}}$, and thus A hypermaximal in $\overline{\mathfrak{M}}$. All φ_t, except for the t-set $\overline{T}_0$ of measure 0, belong to $\overline{\mathfrak{M}}$. Thus we can apply the considerations of §4 to A and to the φ_t in $\overline{\mathfrak{M}}$: a closed linear manifold $\mathfrak{M}$ (contained in $\overline{\mathfrak{M}}$) exists, which has all properties required in Theorem III.

And this completes the proof of Theorem III in its generality.

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[34] We can form $U^{-q}\bar{\omega}_k$ because $\bar{\omega}_k$ belongs to $\mathfrak{G}$. Remember that U is length-preserving, meaning that $\| Uf \| = \| f \|$, which implies $(Uf, Ug) = (f, g)$, compare [22], p. 76, and [11], p. 70, 71.

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